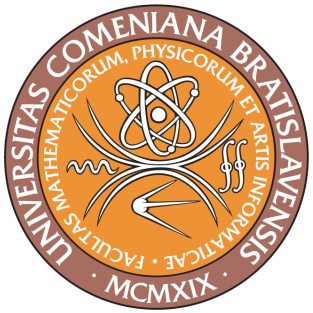
Interplay Between Global and Local Symmetries Computational Aspects

Tatiana Jajcayová

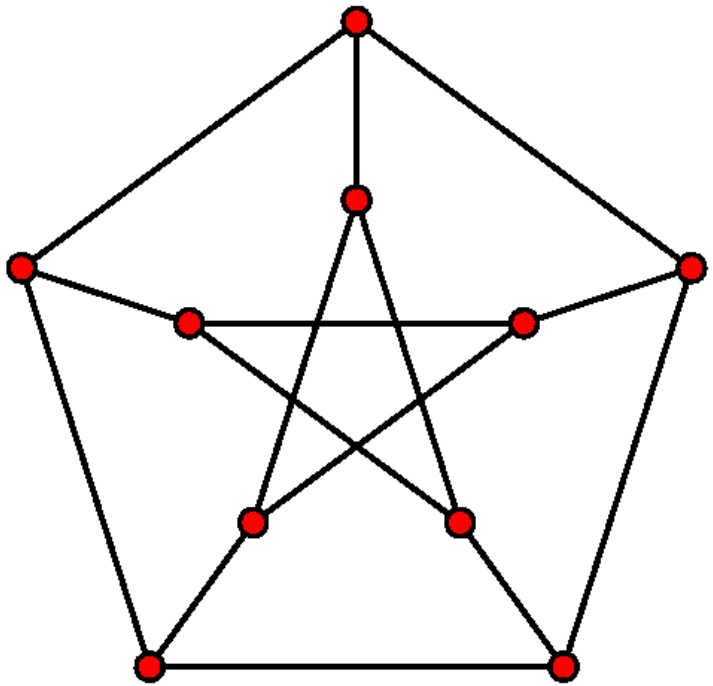


Comenius University, Bratislava

CSD 11
Kranjska Gora, Slovenia
May 4, 2026

Automorphisms vs. Symmetries

An automorphism is an isomorphism from a graph to itself.



The automorphism group captures symmetries of the graph.

Motivation: Symmetries of combinatorial structures

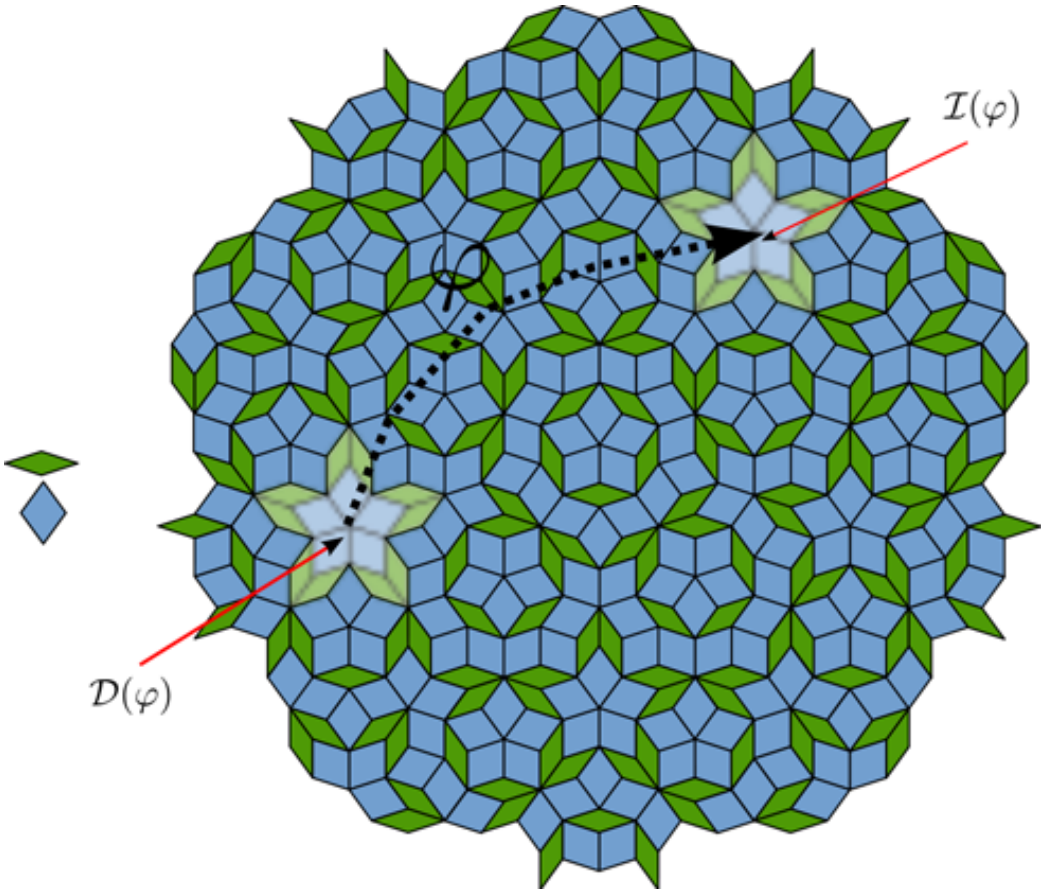
The following problems are polynomially equivalent:

- GI: the graph isomorphism problem
- col-GI: isomorphism problem of colored graphs
- ISO: isomorphism of general combinatorial objects
- Aut(Γ): compute generating set for automorphism group
- $|\mathrm{Aut}(\Gamma)|$: determine the size of Aut(Γ)

The graph isomorphism problem is the problem of detecting symmetries of combinatorial structures.

Partial Symmetries

Penrose tiling



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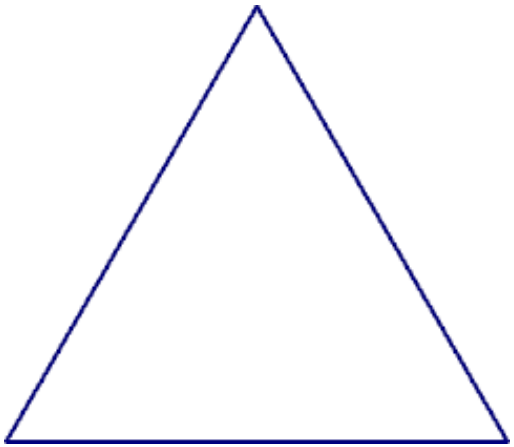
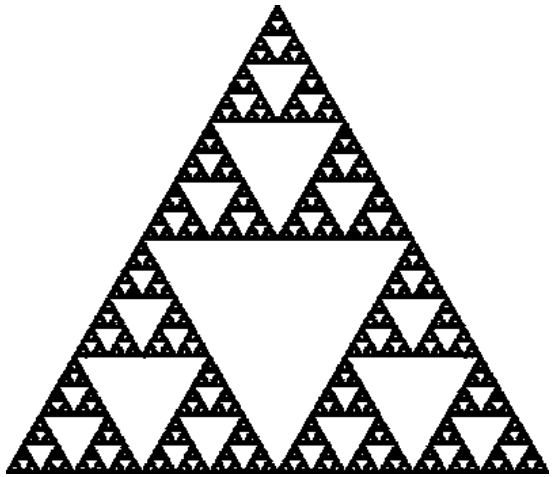
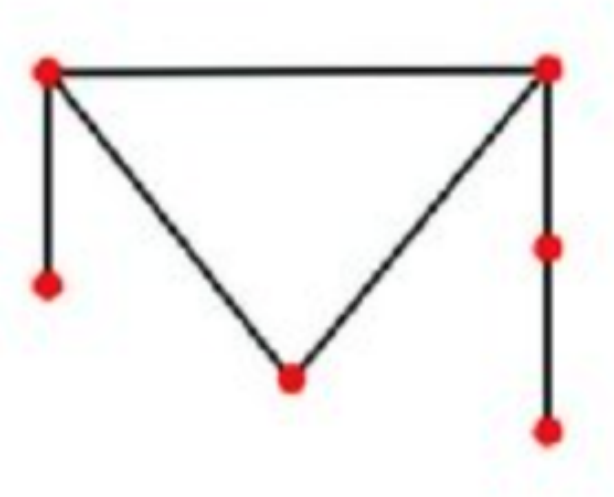




Absence of symmetry

A graph Γ is called *asymmetric* (or rigid) if it does not have non-trivial automorphisms. (i.e. $|Aut(\Gamma)| = 1$).

Example:



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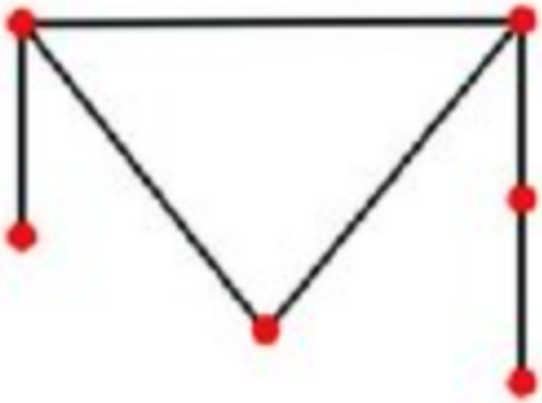

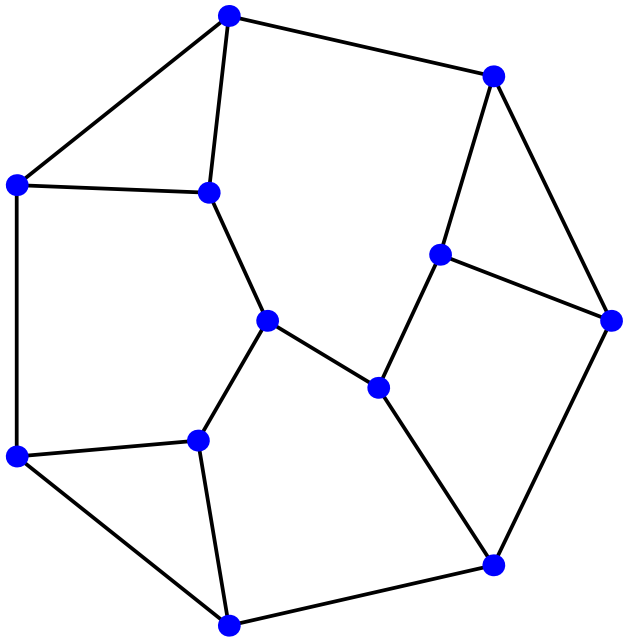

Asymmetric graphs

- (Erdös, Rényi, 1963)
  Almost all graphs are asymmetric

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- in particular, $Aut(\Gamma)$ is trivial for all those graphs

Worst case instances for GI testers

[Neuen, Schweitzer, 2017] Algorithms for testing graph isomorphism can in the worst case have exponential running time.

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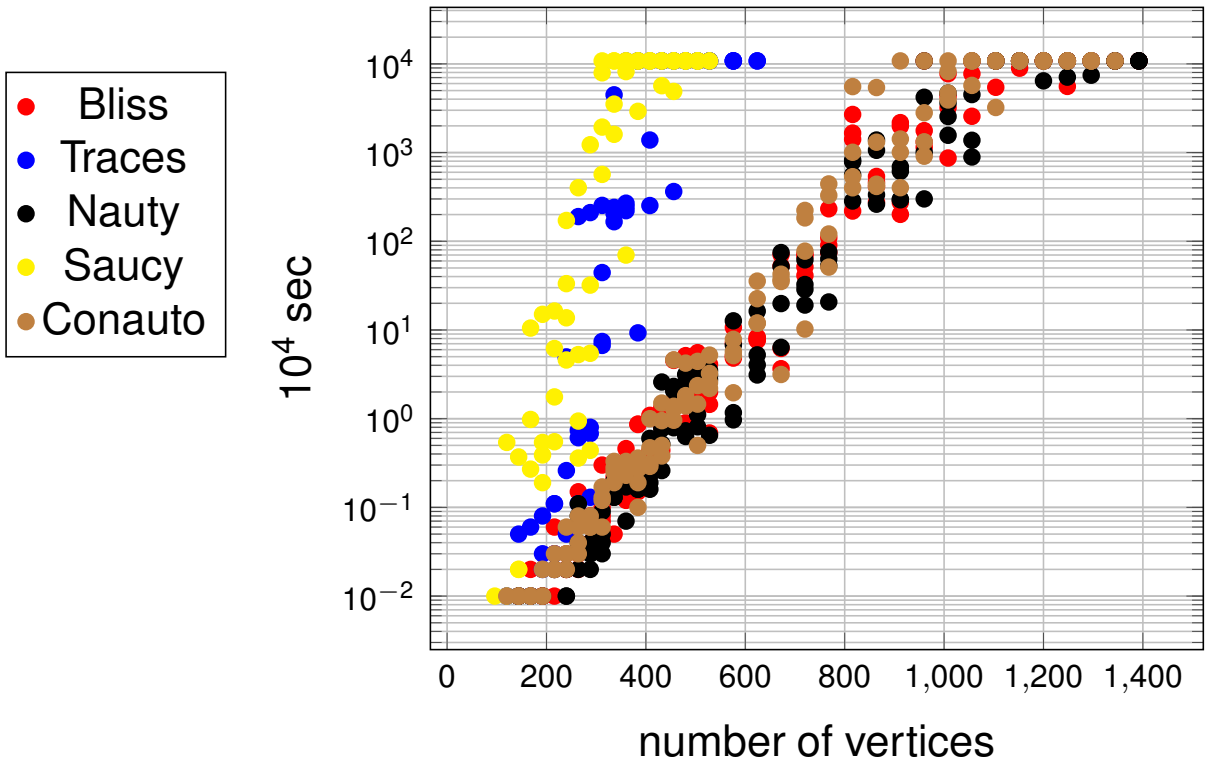


Figure: From (Schweitzer, 2017)

These benchmarks are asymmetric (rigid) graphs.

Asymmetric graphs

Erdös, Rényi:

- Symmetrization: removing (r) and adding (s) edges to make a graph symmetric
- Degree of asymmetry $A(\Gamma)$: the minimum of $r+s$ taken over all possible symmetrizations
- The asymmetry of a graph of order n can not exceed $\frac{n-1}{2}$; and this estimate is asymptotically best possible
- The relative asymmetry of Γ,

$$a(\Gamma) = \frac{A(\Gamma)}{\frac{n-1}{2}}$$

$0 \leq a(\Gamma) \leq 1$

Symmetry vs. Assymetry: Minimal asymmetric graphs

An undirected graph G on at least two vertices is *minimal asymmetric* if G is asymmetric and no proper induced subgraph of G on at least two vertices is asymmetric.

Minimal asymmetric graphs

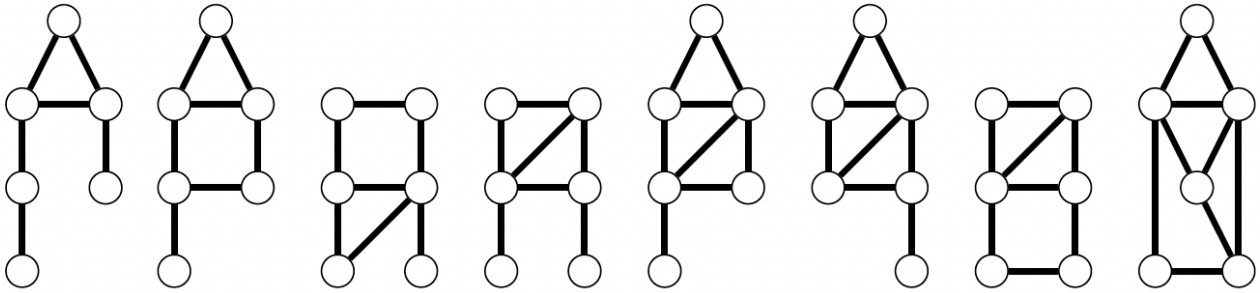
Theorem (Schweitzer, Pascal; Schweitzer, Patrick, 2017)

There are exactly 18 finite minimal asymmetric undirected graphs up to isomorphism.

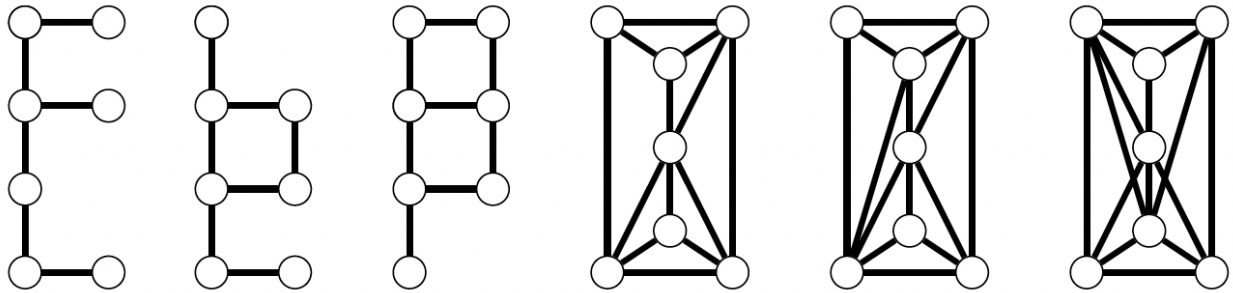
Nešetřil's conjecture: There are exactly 18 minimal asymmetric graphs (coming in 9 complementary pairs).

Nešetřil and Sabidussi earlier established a close connection between minimal asymmetric graphs and minimal involution-free graphs.

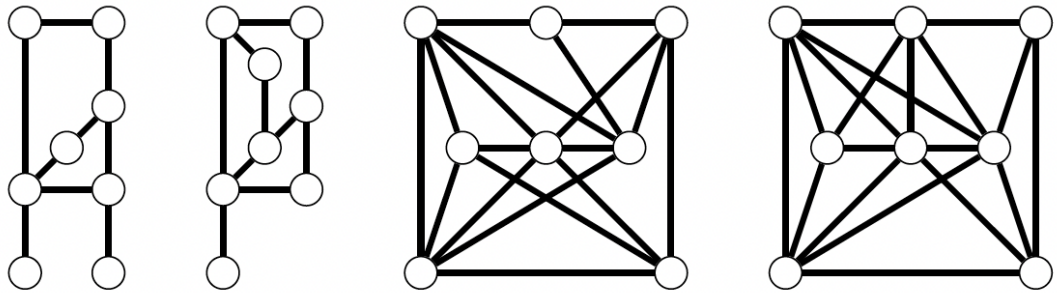
Minimal asymmetric graphs



X_1 $(6, 6, X_8)$ X_2 $(6, 7, X_7)$ X_3 $(6, 7, X_6)$ X_4 $(6, 7, X_5)$ X_5 $(6, 8, X_4)$ X_6 $(6, 8, X_3)$ X_7 $(6, 8, X_2)$ X_8 $(6, 9, X_1)$



X_9 $(7, 6, X_{14})$ X_{10} $(7, 7, X_{13})$ X_{11} $(7, 8, X_{12})$ X_{12} $(7, 13, X_{11})$ X_{13} $(7, 14, X_{10})$ X_{14} $(7, 15, X_9)$



X_{15} $(8, 9, X_{18})$ X_{16} $(8, 10, X_{17})$ X_{17} $(8, 18, X_{16})$ X_{18} $(8, 19, X_{15})$

Symmetry vs. Asymmetry

Cayley graph $\mathcal{C}(G, X)$.

- Cayley graphs are well-known highly symmetric (vertex-transitive) graphs; $G_L \leq Aut(C(G, X))$

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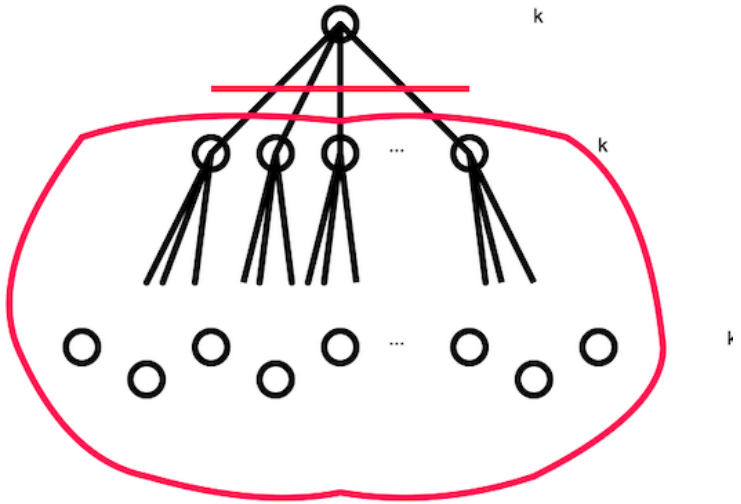

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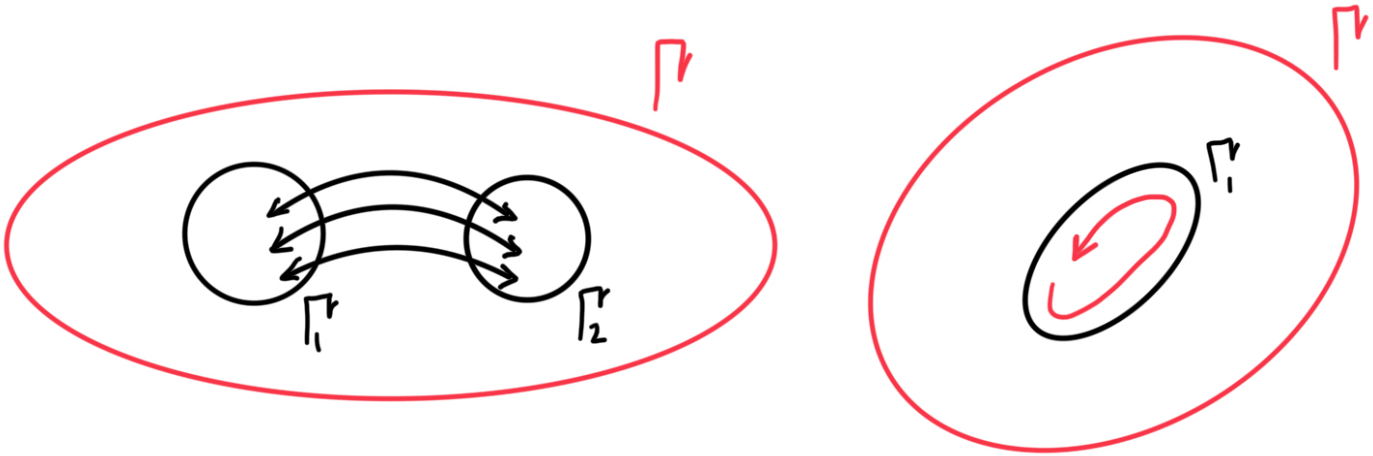

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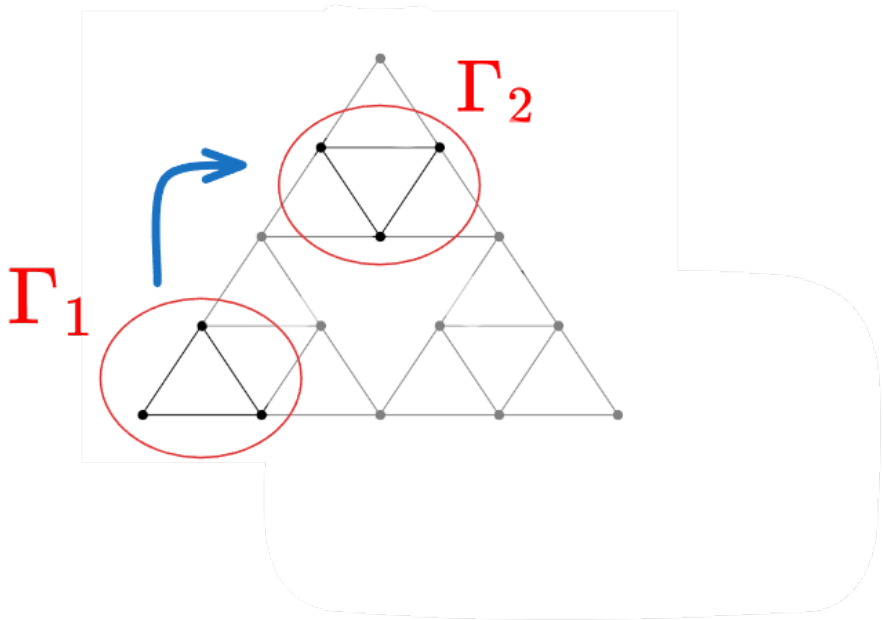
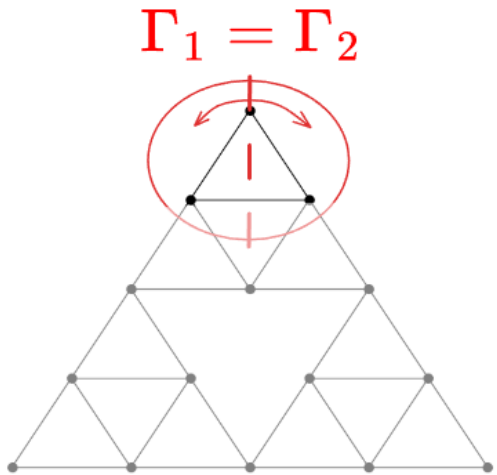
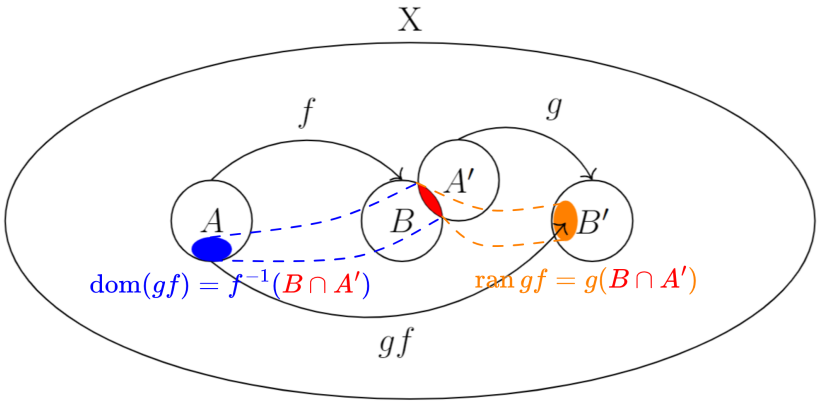




The rank of a partial automorphism is given by the size of its domain.

Inverse Monoid of Partial Automorphisms

The set of all partial automorphisms, denoted PAut(Γ) with the composition and partial inverse of partial maps forms an inverse monoid.



$\mathrm{PAut}(\Gamma) \leq \mathrm{PSym}(V)$

Inverse Monoid

A set together with an associative binary operation is called a *semigroup*

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A monoid M is called **inverse**

- if for every $a \in M$ there exists a unique element a^{-1} s.t.

$$a \cdot a^{-1} \cdot a = a$$

$$a^{-1} \cdot a \cdot a^{-1} = a^{-1}$$

"Archetypal" inverse semigroup PSym(X)

PSym(X) - set of all partial permutations of X
= bijections between subsets of X (including $\emptyset$).

$$\varphi : Y \to Z \qquad Y, Z \subseteq X$$

Y - domain $dom\varphi$
Z - range $ran\varphi$

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The cycle notation of classical permutations generalizes by the addition of a notion called a path, which (unlike a cycle) ends when it reaches the "undefined" element.

$$\mathrm{dom}(x_1, x_2 ... x_k] = \{x_1, x_2, ..., x_{k-1}\}$$

$$\mathrm{ran}(x_1, x_2 ... x_k] = \{x_2, x_3, ..., x_k\}$$

PSym(X)

- Operation on PSym(X) - is composition of partial maps:
  Let α and β be partial permutations of a set X; α and β can be composed (from left to right) on the largest domain upon which it "makes sense" to compose them (may be the $\emptyset$)

$$\operatorname{dom} \alpha\beta = [\operatorname{im} \alpha \cap \operatorname{dom} \beta]\alpha^{-1}$$

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- Local Identities: $A \subset X$ - id_A, idempotents
- Zero - PSym(X) has also zero element - empty map - $id_\emptyset$

Wagner-Preston representation

While groups can be represented as symmetries:

Theorem (Cayley)

Every (finite) group can be represented as a group of permutations of a (finite) set.

Inverse semigroups can be represented as partial symmetries:

Theorem (Wagner-Preston)

Every (finite) inverse semigroup can be represented as the inverse semigroup of partial bijections of a (finite) set.

Restrictions vs. extensions

It is clear that all restrictions of an automorphism of a graph are partial automorphisms.

But not all partial automorphisms extend to a (global) automorphism.

Homogeneous graphs

A graph Γ is *homogeneous* if any isomorphism between induced subgraphs extends to an automorphism of Γ.

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Theorem (Gardiner 1976)

A finite homogeneous graph is one of the following:

- *a disjoint union of complete graphs of the same size*
- *a regular complete multipartite graph*
- *the 5-cycle*
- *the line graph of $K_{3,3}$*

Extensions of partial automorphisms

Jaroslav Nešetřil, Matěj Konečný, ...

Extension property for partial automorphisms, EPPA

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Let A be a structure and let B be its (induced) substructure. A is an EPPA-witness for B if every partial automorphism of B extends to an automorphism of A.

A class $\mathcal{C}$ of **finite** structures has EPPA if for every $B \in \mathcal{C}$ there is $A \in \mathcal{C}$, which is an EPPA-witness for B.

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Theorem (Hrushovski 1992)

The class of all finite graphs has EPPA.

Inverse Monoid of Partial Automorphisms of Graphs

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- Jajcay, Jajcayová, Szakács, and Szendrei (2021) gave a characterization of partial automorphism inverse monoids for graphs.

Structure of inverse monoids

Green's relations:

$s, t \in S$. We define $\mathcal{L}$ and $\mathcal{R}$:

- $s \mathcal{L} t \Leftrightarrow \exists\, x, y \in S$ s.t. $xs = t$ & $yt = s$,
- $s \mathcal{R} t \Leftrightarrow \exists\, x, y \in S$ s.t. $sx = t$ & $ty = s$.

In $PSym(X)$,

$$\varphi_1 \mathcal{L} \varphi_2 \Leftrightarrow \operatorname{dom} \varphi_1 = \operatorname{dom} \varphi_2,$$
$$\varphi_1 \mathcal{R} \varphi_2 \Leftrightarrow \operatorname{ran} \varphi_1 = \operatorname{ran} \varphi_2.$$

Structure of PAut(Γ) for graph Γ

- finding the whole monoid is hard (both in time and memory)

Proposition (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

For any graph Γ, *the* $\mathcal{D}$*-classes of* PAut(Γ) *correspond to the isomorphism classes of induced subgraphs of* Γ, *that is, two elements are* $\mathcal{D}$*-related if and only if the subgraphs induced by their respective domains (or images) are isomorphic.*

Partial order for $\mathcal{D}$-classes: "subgraph" relation

Example

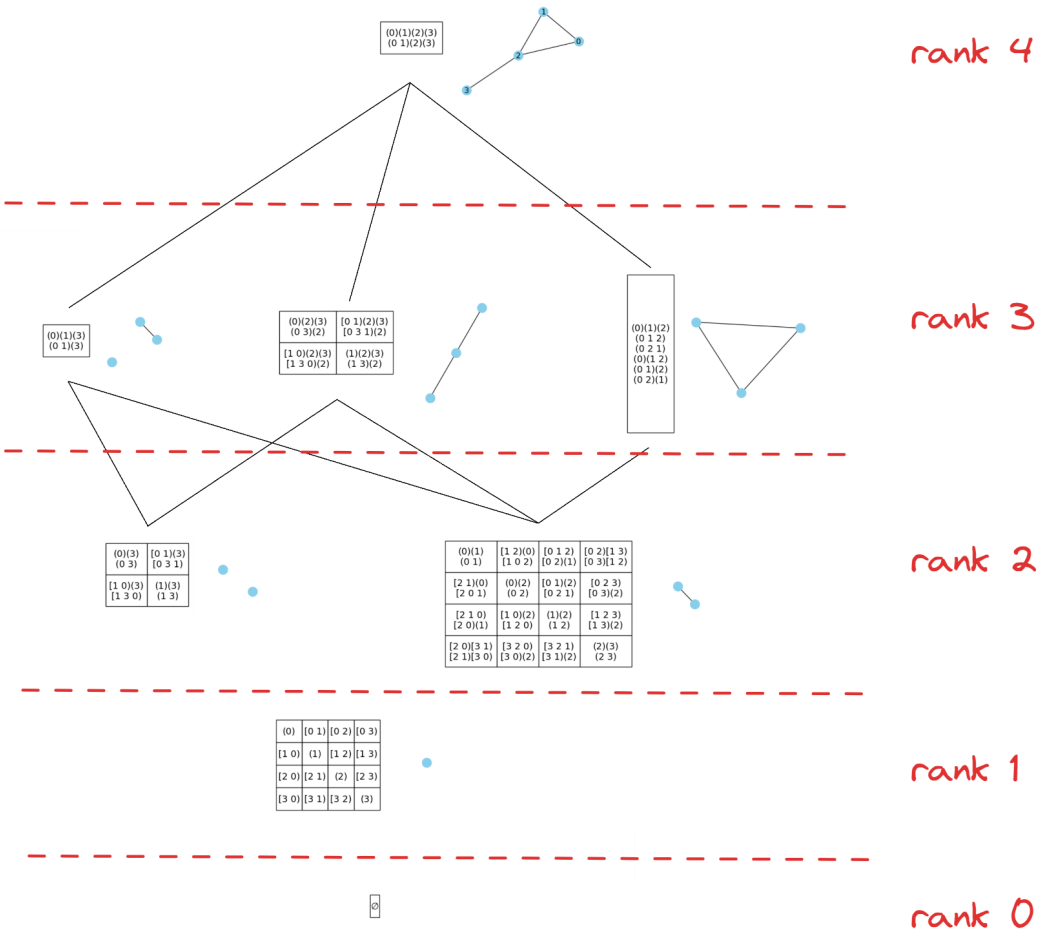

Figure: The Green-class structure of partial graph automorphisms

Asymmetric depth & Symmetry Level of a Graph

Definition

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The **level of symmetry** of a graph Γ of order n is the ratio between the largest rank of a non-trivial partial automorphism of Γ and its order n.

- A graph Γ admitting a non-trivial automorphism has level of symmetry 1

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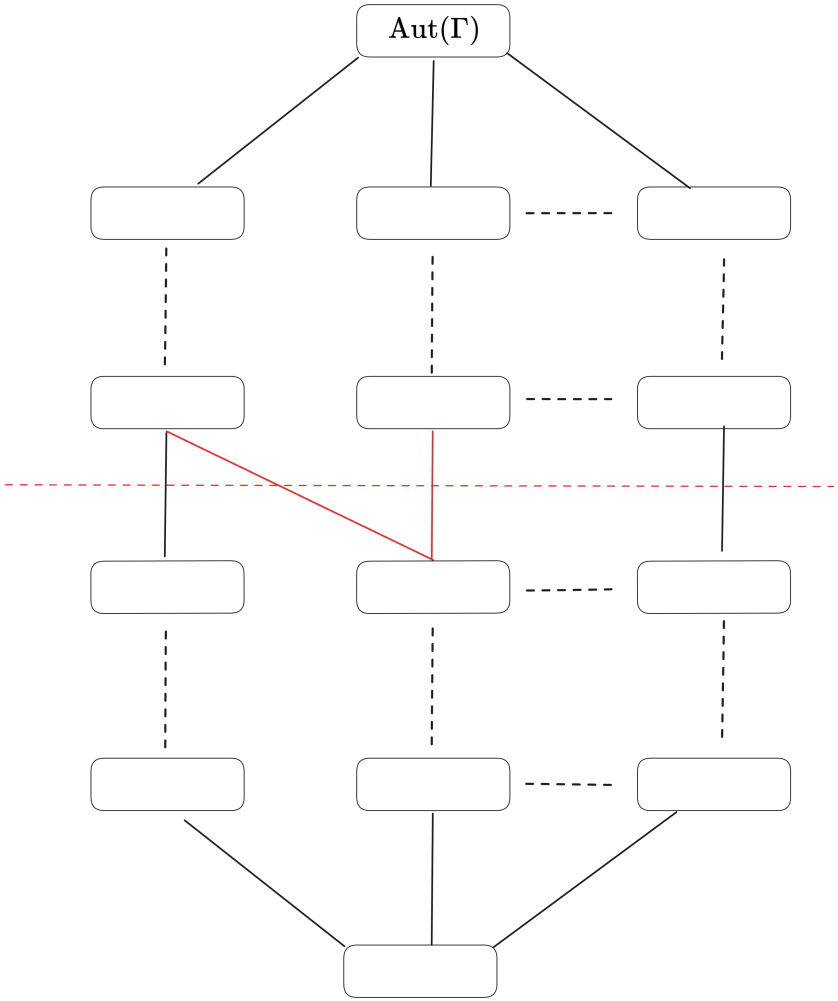
- The level of symmetry of Γ is equal to the level of symmetry of its complement

Asymmetric depth



$\rightarrow$ rank n

$\vdots$

$\rightarrow$ rank $n - i$

$\vdots$

$\rightarrow$ rank k_{max}

Asymmetric depth & Symmetry Level of a Graph

Questions:

- Given $k \geq 1$, does there exist a graph Γ of order n with level of symmetry equal to $\frac{n-k}{n}$?
- What is the minimal level of symmetry of a graph Γ of order n as a function of n?
- ...

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(Gál, Mériová, Cingel, Pastorek, 2023+)
Searches, tests, computational results, bounds,...

Is there an upper bound for asymmetric depth $d = n - k_{max}$?

(with Jan Pastorek)

We proved that every n-vertex simple graph Γ admits a nontrivial partial automorphism on more than half of its vertices:

$$S(\Gamma) \geq 1 - \frac{n-1}{2n} > \frac{1}{2} \quad \text{equivalently} \quad d(\Gamma) \leq \frac{n-1}{2}.$$

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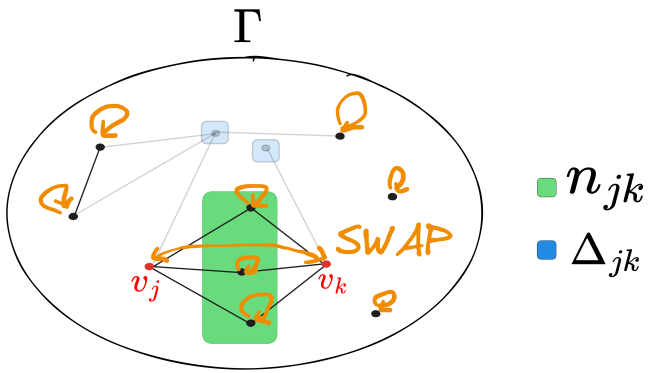
Idea of the proof:

- Define a partial automorphism that will *swap just two vertices*
- Δ_{jk} - symmetric difference of the neighborhoods of v_j and v_k
- For any vertices $v_j \neq v_k$



This bound is sharp

Theorem

The graph Γ with 37 vertices with the following graph6 string representation meets the general bound

```
d~~vVB`pTErceQcmqK[YlafME|CLLZHTTWplTc[ie[
    ViMopRcqpQwNHhFyIqMXirMFJXKuBtFXipSmPdYozKkiVpbIxWcm`UclwXcmjLOjFRpIsl
```

Theorem

The graph Γ from the previous theorem with 37 vertices is of the smallest possible order that meets the bound.

This bound is sharp

- Computationally difficult - in both time and memory
- Suprisingly, the "most asymmetric" graph meets many regularitu conditions and it is a strongly regular graphs with parameters: $(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}) \rightarrow$ conference graphs
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- more in the next talk

Isomorphism Problem for Asymmetric Graphs

- In view of the prevalence of asymmetric graphs within the class of all graphs and the fact that graphs 'causing problems' to some of the most widely used algorithms are usually highly structured, it is interesting to restrict some of the isomorphism questions to asymmetric graphs, i.e., to consider the Asymmetric Graph Isomorphism

  Problem of deciding whether two given asymmetric graphs are isomorphic

Isomorphism Problem for Asymmetric Graphs

- In view of the prevalence of asymmetric graphs within the class of all graphs and the fact that graphs 'causing problems' to some of the most widely used algorithms are usually highly structured, it is interesting to restrict some of the isomorphism questions to asymmetric graphs, i.e., to consider the Asymmetric Graph Isomorphism

  Problem of deciding whether two given asymmetric graphs are isomorphic

- Specifically, one could ask whether deciding the Asymmetric Graph Isomorphism Problem might be easier than the general Graph Isomorphism Problem

What is easier?

(Schweitzer, Arvind, Das, Mukhopadhya, 2017)
For tournaments finding all symmetries and detecting asymmetry are polynomially equivalent.

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- A possible way to answering this problem would lead through designing a polynomial complexity (in the order of the input graph) construction that would associate an asymmetric graph $\tilde{\Gamma}$ to each graph Γ while satisfying the property:

$$\tilde{\Gamma} \cong \tilde{\Gamma'} \text{ if and only if } \Gamma \cong \Gamma'$$

Isomorphism Problem for Asymmetric graphs

- Using the asymmetry level of a graph might allow for a further refinement of this question where one could ask whether the Asymmetric Graph Isomorphism Problem becomes easier with an increasing level of asymmetry of the considered graphs

Fixed Point Free Automorphism Problem

- Another complexity problem that appears to carry some relevance with regard to the above problems is the

  Fixed Point Free Automorphism Problem:
  Given a graph Γ, determine whether there exists a fixed-point-free automorphism of Γ
  (automorphism φ for which $\varphi(u) \neq u$, for all $u \in V(\Gamma)$)

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- and remains NP-complete even when restricted to split, bipartite, k-subdivided and H-free graphs (provided H is not an induced subgraph of P_4) by the resent results of Aida Abiad et al. in 2026+

Advocacy for studying $PAut(\Gamma)$

- partial automorphisms capture local structure and local symmetries of a graph
- Babai (2018) argued that solving Graph Isomorphism Problem (GI) will involve looking at local structures of graphs (and so possibly partial automorphisms)
- the fastest procedure that helps to decide Graph Isomorphism problem (Weisfeiler-Leman algorithm) is based on the idea of local symmetries and aggregation of local information
- distinguishes structures inaccessible to group theory

Structure of $PAut(\Gamma)$ - inspiration from groups

Which groups are $Aut(\Gamma)$?

Theorem (Frucht 1939)

For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$.

When is an inverse monoid of partial permutations the partial automorphism monoid of a graph?

Situation in Inverse monoids is much more restrictive:

Theorem (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Given an inverse submonoid $S \leq \mathrm{PSym}(X)$, where X is a finite set, there exists a graph with vertex set X whose partial automorphism monoid is S if and only if the following conditions hold:

1. *S is a full inverse submonoid of $\mathrm{PSym}(X)$,*
2. *for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A,*
3. *the rank 2 elements of S form at most two $\mathcal{D}$-classes,*
4. *the $\mathcal{H}$-classes of rank 2 elements are nontrivial.*

When is an (abstract) inverse monoid *isomorphic* to the partial automorphism monoid of a graph?

Theorem (R.Jajcay, T.J., N. Szakács, M. Szendrei 2021)

Given a finite inverse monoid S, there exists a finite graph whose partial automorphism monoid is isomorphic to S if and only if the following conditions hold:

1. *S is Boolean,*
2. *S is fundamental,*
3. *for any subset $A \subseteq S$ of compatible 0-minimal elements, if all 2-element subsets of A have a join in S, then the set A has a join in S,*
4. *S has at most two $\mathcal{D}$-classes of height 2,*
5. *the $\mathcal{H}$-classes of the height 2 $\mathcal{D}$-classes of S are nontrivial.*

Algorithmic aspects of determining $PAut(\Gamma)$

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- Given an inverse monoid of partial automorphisms of Γ, if one wishes to determine whether it is the complete $PAut(\Gamma)$, one should ask whether it satisfies the "JJSS" characterization
- A partial automorphism φ of Γ is **complete** if there exists no partial automorphism ψ of Γ that is an extension of φ

Constructing PAut(Γ) recursively

Let $\Gamma = (V, E)$ be a graph,

- the results on symmetry levels of arbitrary graphs prove the existence of complete or extendable partial automorphisms of rank at least $\frac{n+1}{2n}$ for all graphs of order n

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- the results on symmetry levels of arbitrary graphs prove the existence of complete or extendable partial automorphisms of rank at least $\frac{n+1}{2n}$ for all graphs of order n
- if $\varphi \in PSym(V)$ is a partial permutation of rank at least 2, then $\varphi \in PAut(\Gamma)$ if and only if $\varphi|_Y \in PAut(\Gamma)$ for any 2-element subset Y of dom φ, which means that we can iteratively build the whole inverse monoid of a graph from its partial automorphisms of rank 2

Constructing PAut(Γ) recursively

- we say that two partial automorphisms $\varphi_1, \varphi_2 \in PAut(\Gamma)$ are *compatible* if they are both restrictions of some partial automorphism $\psi \in PAut(\Gamma)$, i.e., if they agree on dom $\varphi_1 \cap$ dom φ_2 and have no shared image vertices on the rests of their domains

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- thus, two partial automorphisms $\varphi_1, \varphi_2 \in PAut(\Gamma)$ of rank 2 are compatible if and only if both their domains and ranges are disjoint or their domains share a vertex on which they agree and the images of their non-shared domain vertices are different (we shall call the checking for this property a *compatibility comparison*)

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4. stop when no partial automorphisms of rank i are compatible with any rank 2 partial automorphisms (i.e., when all partial automorphisms of rank i are complete)
5. alternately, to obtain all partial automorphisms of rank $i = i' + i''$ from all rank i' and rank i'' partial automorphisms, combine each compatible pair into a rank $i = i' + i''$ partial automorphism

Computational complexity of the algorithm

- the algorithm constructs the entire $\mathrm{PAut}(\Gamma)$ in at most $n = |V(\Gamma)|$ steps, each of which consists of $|(\mathrm{PAut}(\Gamma))_i| \cdot |(\mathrm{PAut}(\Gamma))_2|$ compatibility comparisons (where $|(\mathrm{PAut}(\Gamma))_i|$ stands for the number of partial automorphisms of Γ of rank i)

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- Determining whether a partial automorphism is complete and forming all its extensions of rank increased by 1 requires $|(\mathrm{PAut}(\Gamma))_2|$ compatibility comparisons
- The problem of deciding whether a given partial automorphism can be extended into a (full) automorphism is polynomially equivalent to the Graph Isomorphism Problem
- Similarly, the problem of finding a full automorphism for a given restriction (if it exists) is also polynomially equivalent to the Graph Isomorphism Problem

Orbits of the automorphism group

The algorithm can also be used to compute the *pre-orbits* (unions of orbits) of the action of the automorphism group of Γ on $V(\Gamma)$:

- for each $(\mathrm{PAut}(\Gamma))_i$, define on $V(\Gamma)$ an equivalence relation $\sim_i$ by setting $u \sim_i v$ if and only if there exists a $\varphi \in (\mathrm{PAut}(\Gamma))_i$ mapping u to v

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  - hence, if their exists an i, $2 \leq i \leq |V(\Gamma)|$, for which u and v are not $\sim_i$ equivalent, then u and v belong to different orbits of the automorphism group of Γ
  - equivalently, the equivalence classes of each $\sim_i$ are unions of the orbits of the automorphism group of Γ on $V(\Gamma)$

Weisfeiler-Leman algorithm

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- The Weisfeiler-Leman algorithm is also used to determine the pre-orbits of $Aut(\Gamma)$
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- If no changes occurred in the refinement step, the pre-orbits of $Aut(\Gamma)$ are determined by different colors
- (k-WL) can be computed for different values of k (with each higher value of k giving a refinement of the previous pre-orbits, until the pre-orbits become stable)
- Stable coloring by k-WL can be computed in time $O(n^{k+1} \log n)$

Weisfeiler-Leman algorithm

- k-WL **distinguishes** G and H if there is a colour c of the stable colouring such that G and H have different numbers of k-tuples of colour c.
- k-WL **identifies** G if it distinguishes G from all other graphs.

Remarks on k-WL

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- For every k, there are graphs, which k-WL does not identify.
- Interesting (for theoretical and practical reasons) cases are for $k \leq 3$:
  - Used in practice
  - 1-WL identifies all forests (Immerman, Lander 1990).
  - 1-WL identifies almost all graphs (Babai, Erdös, Selkow 1980).
  - 1-WL and 2-WL do not distinguish any two regular graphs with the same number of vertices and the same number of edges.
  - 3-WL identifies almost all regular graphs (Bollobás 1982)
  - It is decidable in $O((n+m)logn)$ time whether 1-WL identifies a given graph (Kiefer et al., 2015)
  - Similar result for $k \geq 2$ is **unlikely** (Schweitzer, Kiefer, 2020)

Comparing the two algorithms

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Comparing the two algorithms

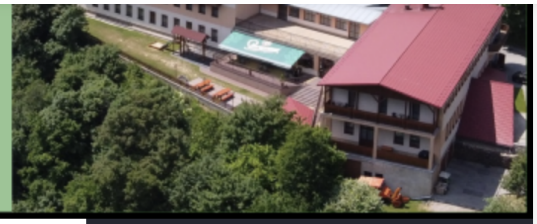
- both algorithms allow for computing pre-orbits of $Aut(\Gamma)$
- Calculations required for the i-WL and calculations required to compute $(\mathrm{PAut}(\Gamma))_i$ are somewhat comparable
- the complexity of i-WL increases with increasing i and then starts decreasing and so does the complexity of calculating $(\mathrm{PAut}(\Gamma))_i$
- both algorithms ultimately determine the orbits of $Aut(\Gamma)$; where stopping the algorithm using partial automorphisms before reaching the order of Γ not only determines that the automorphism group is trivial (and therefore each vertex is its own orbit), but also determines the level of symmetry of Γ

Thank you for listening!

Upcoming workshop in Slovakia

Upcoming workshop in Slovakia

ITAT Hotel Vršatec, Biele Karpaty



Invited speakers

Jorik Jooken (KU Leuven, Belgium): *How can computers help in obtaining new results in graph theory?*

This talk consists of two parts. In the first part, I will give an introduction to the domain of computer-assisted graph theory, which is concerned with developing algorithms in order to help researchers gain insights into various graph theoretical questions. In particular, I will discuss a broad range of popular techniques from this domain and briefly talk about their applications. In the second part, I will zoom in on an algorithm for exhaustively generating graphs and show how executing this algorithm leads to

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