Extremely asymmetric graphs and partial automorphisms

How far from having a symmetry can a graph be?

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Joint work with Tatiana Jajcayová
FMFI, KAI, Comenius University in Bratislava, Slovakia

CSD 2026

Partial automorphisms for local symmetries

Definition (Partial automorphism)

Partial automorphism of a graph $\Gamma = (V, E)$ *is an isomorphism between two induced subgraphs* Γ_1, Γ_2 *of* Γ.

- Rank of a partial automorphism is the order of its domain.

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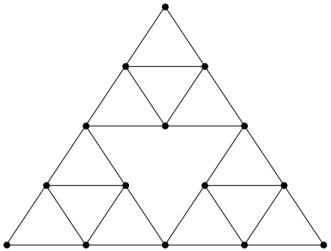

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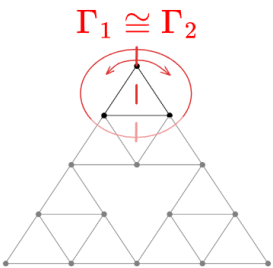

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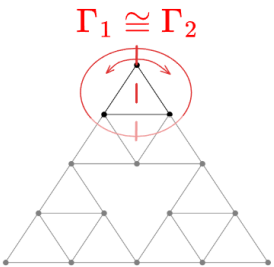

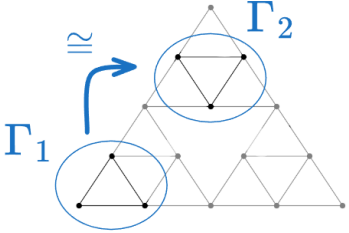

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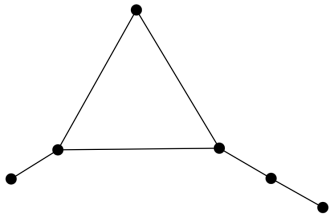
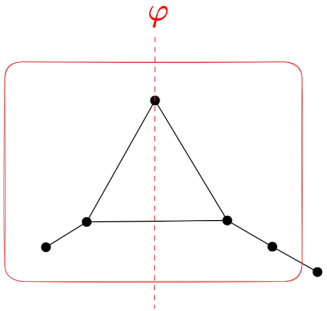
Definition (Asymmetric depth (Cingel, Jajcayová, and Pastorek, 2024))

Let $k_{\max}$ be the rank of the largest nontrivial partial automorphism of Γ on n vertices.

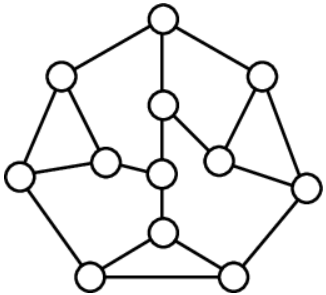
$$d(\Gamma) := n - k_{\max}.$$



- $n = 6$



- $n = 6$
- $k_{\max} = 5$
- $d = 1$



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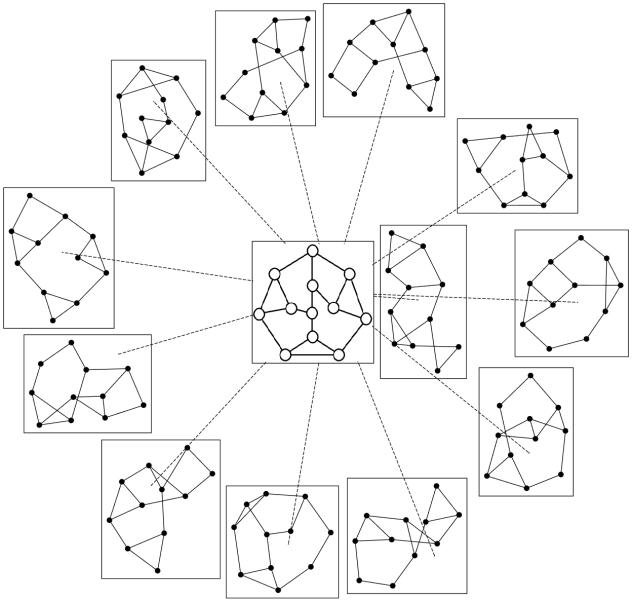

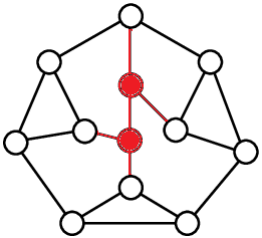

- $n = 12$
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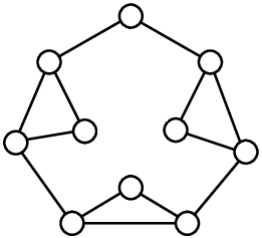

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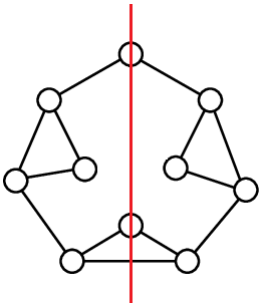

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3. Can we obtain sharper bounds for special graph classes? (planar)

Idea of the proof

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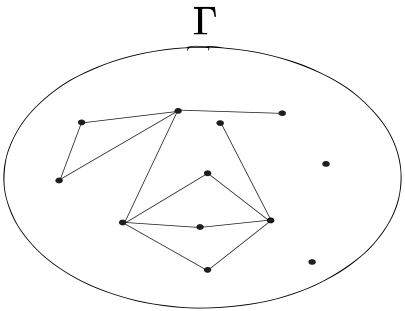
- Define a partial automorphism that will *swap just two vertices*



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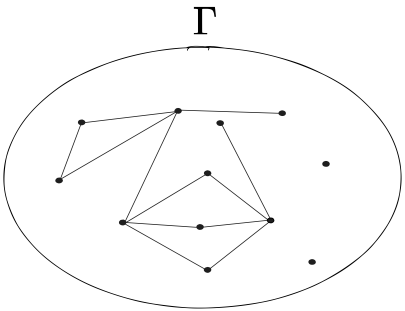

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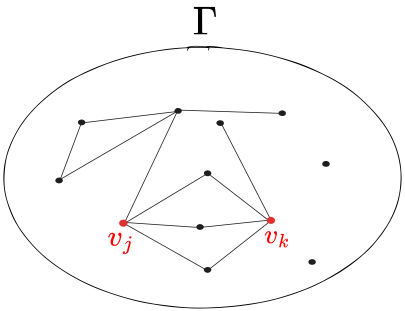

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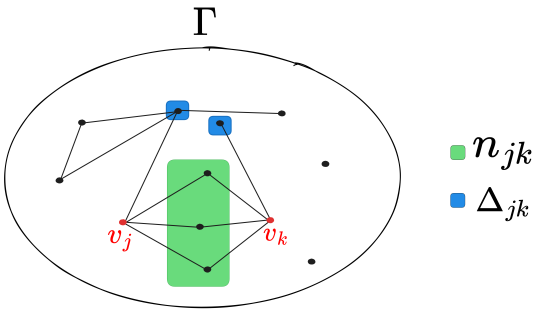

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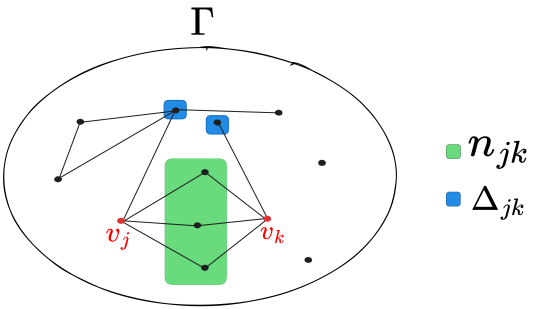

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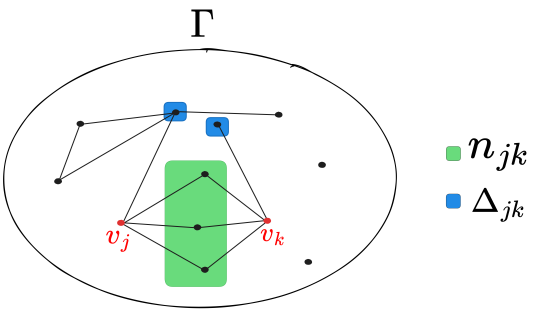
- $d \leq |\Delta_{jk}|$



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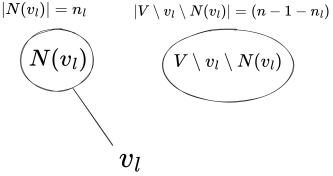

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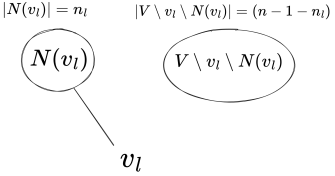
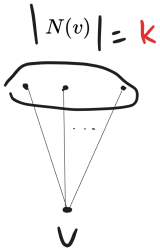
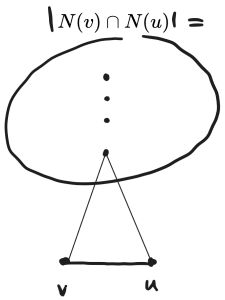
Using $n_\ell(n-1-n_\ell) = \left(\frac{n-1}{2}\right)^2 - \left(n_\ell - \frac{n-1}{2}\right)^2$ and ...

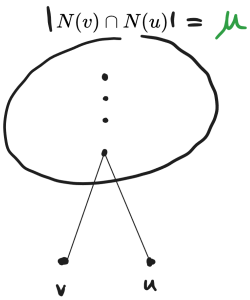


Strongly regular graphs

Strongly regular graph (SRG) is a regular graph $\Gamma = (V, E)$ with n vertices and degree k such that for some given integers $\lambda, \mu \geq 0$







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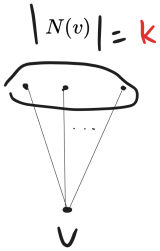

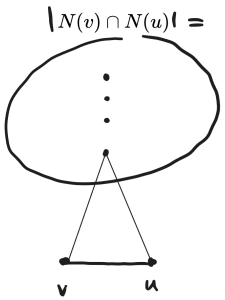

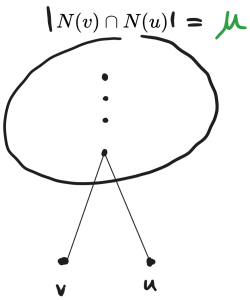

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Conference graphs

$$\mathrm{srg}(n, \underbrace{\tfrac{n-1}{2}}_{k}, \underbrace{\tfrac{n-5}{4}}_{\lambda}, \underbrace{\tfrac{n-1}{4}}_{\mu})$$

5-cycle graph

generalized quadrangle (2,1)

13-Paley graph

17-Paley graph

25-Paley graph

29 Paley graph

37-Paley graph

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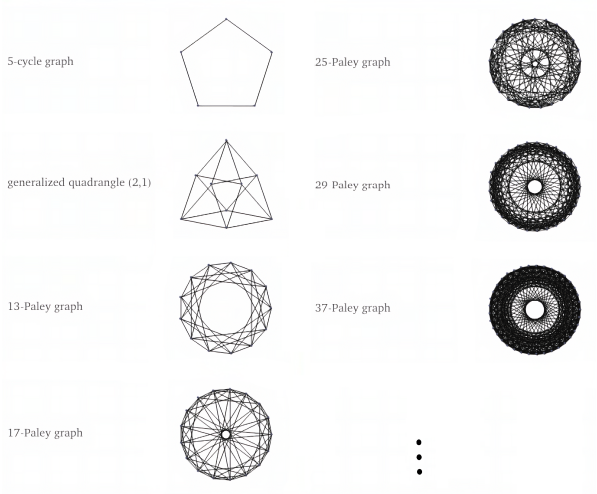

Theorem 1 (2026+)

Let Γ be a graph on n vertices. If $d(\Gamma) = \frac{n-1}{2}$, then Γ is a conference graph.

Tight bound: conference graphs

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n	5	9	13	17	25	29	37
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🧐 If there was a nontrivial partial automorphism of higher rank than 20, then there is its restriction - nontrivial partial automorphism at rank 20.

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💪 I want to check them in parallel but 😟

- Problem 2 (space): if we are encoding graph in canonical hash of $s = 64$ bits
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  - Bloom filter - Monte Carlo $\rightarrow$ fast probabilistic answer

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  - IO slowdown

Prefilter - SAT decision: does $d(\Gamma) \le n - k$?

Variables. $x_{u,v} \equiv [\varphi(u) = v]$, $dom_u \equiv [u \in \operatorname{dom}\varphi]$ $O(n^2)$ Booleans

(C1) $dom_u \leftrightarrow \bigvee_v x_{u,v}$, $\sum_v x_{u,v} \le 1$ (function)

(C2) $\sum_u x_{u,v} \le 1$ (injective)

(C3) $\neg x_{u,v} \vee \neg x_{u',v'}$ whenever $[uu' \in E] \neq [vv' \in E]$ (edge-pres.)

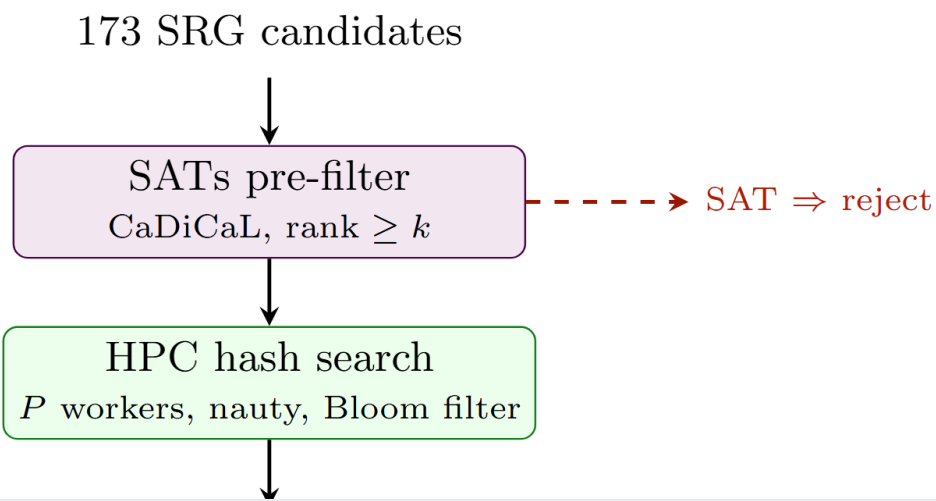
(C4) $\bigvee_{u \neq v} x_{u,v}$ (nontrivial)

(C5) $\sum_u dom_u \ge k$ (rank $\ge k$)

SAT $\iff$ $d(\Gamma) \le n - k$.

Conference graph $\operatorname{srg}(37, 18, 8, 9)$**:** $\sim$1,400 vars, $\sim 4.5 \times 10^5$ clauses

Computational pipeline

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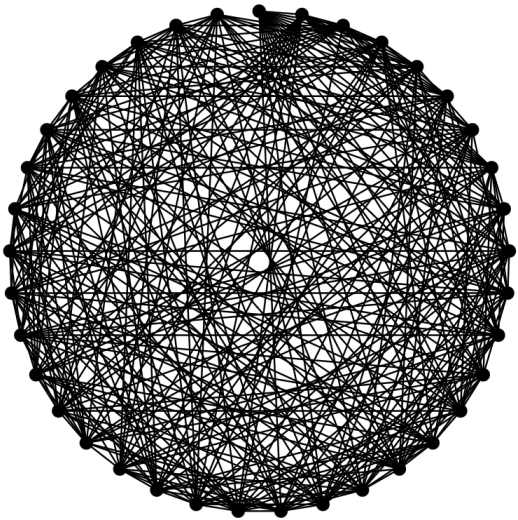
- → HPC Clara at FMFI with 768 GB RAM and 32 threads compute nodes
- ≈ 5 weeks of computations in parallel on two compute nodes



Γ

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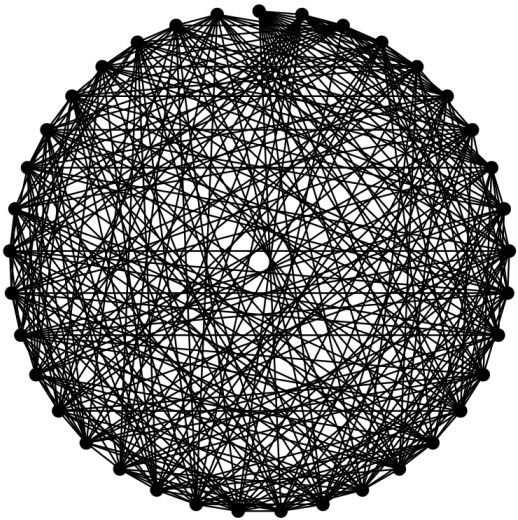

Γ

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Corollary 4 (2026+)

There is a set of 37 + 37 (for complement) asymmetric graphs on 36 vertices that have asymmetric depth of 17 attaining the bound for the even case.

Planar graphs have depth at most 5

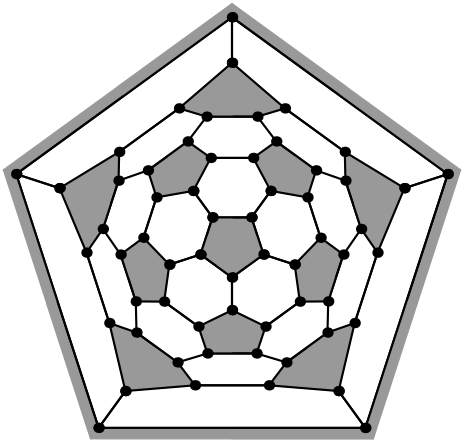
Theorem 5

Every planar graph satisfies $d(\Gamma) \leq 5$.

Proof:

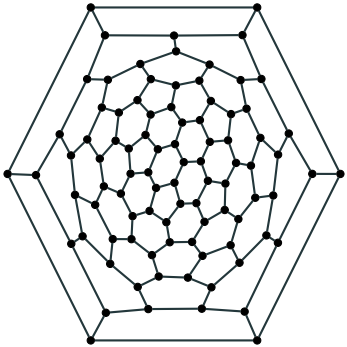
1. For any u, v: $d(\Gamma) \leq \min_{u \neq v} |\Delta_{uv}|$ (local transposition)
2. Aksionov (discharging): every planar graph has a pair u, v with $|\Delta_{uv}| \leq 5$
   $\Rightarrow d(\Gamma) \leq 5$

We have a couple of structural results leading us to duals of IPR fullerenes as the good candidates for planar graphs with depth 5:



Smallest fullerene dual attaining $d = 5$

Using buckygen (Goedgebeur and McKay, 2015) and some optimizations, we found that the dual of the IPR fullerene on 90 vertices shown below is one of the two smallest graphs attaining asymmetric depth 5.



Structural Insights that Cut Down Computations

Unique-extension lemma

In a cubic graph F, any isomorphism $F - u \to F - v$ extends *uniquely* to a full automorphism.

$\Rightarrow$ An asymmetric fullerene has **no** nontrivial partial automorphism of rank $n - 1$.

$\Rightarrow$ $d(F) \geq 2$.

$\Rightarrow$ cubic graphs have $\min_{u \neq v} |\Delta_{uv}| \leq 4 \ \rightarrow \ d(F) \leq 4$.

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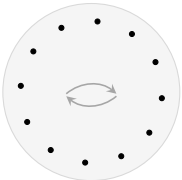
n	Asym. IPR	d=2	d=3	d=4
84	1	0	1	0
86	6	0	2	4
88	11	0	2	9
90	16	0	4	12
92	38	3	1	34
94	89	4	6	79
100	336	2	24	310
106	1054	10	38	1006
110	2111	16	54	2041
118	7670	14	110	7546

Hidden Almost-Global Symmetries

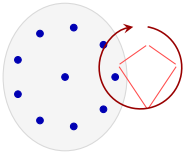
Let φ be a nontrivial partial automorphism of asymmetric IPR F with $(\varphi) = n - k$, $k \leq 4$. Then it is not "trivial", it moves quite a few vertices.

(T1) Local transposition



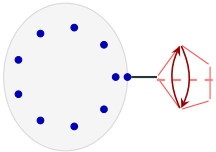
two locally-equivalent vertices swap; rest fixed

(T2) Small component



$\mathrm{dom}(\varphi)$ disconnected; $\varphi = \mathrm{id}$ on big piece

(T3) Almost-separated face



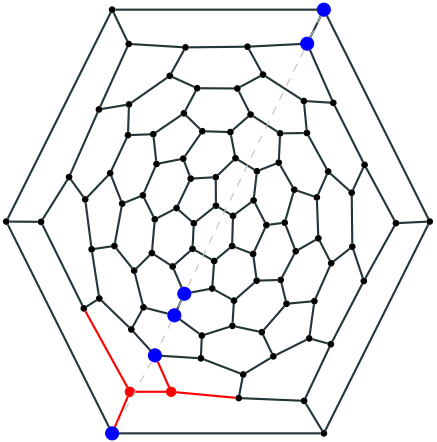
face joined to rest by a single edge; reflected

Interpretation: IPR fullerenes with $d(F) \in \{2, 3\}$ (the rare cases in the table) — possess a *near global symmetry* that is broken only by a tiny local patch of 2–3 vertices.

Smallest IPR fullerene with $d = 2$: 92 vertices.

Blue $=$ axis vertices fixed by PAut;

Red $=$ omitted from domain.



🚴 Thank you for your attention! 🚴

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