How symmetric do you wish your graph to be?

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RECOVERY AND RESILIENCE PLAN

Plan for today

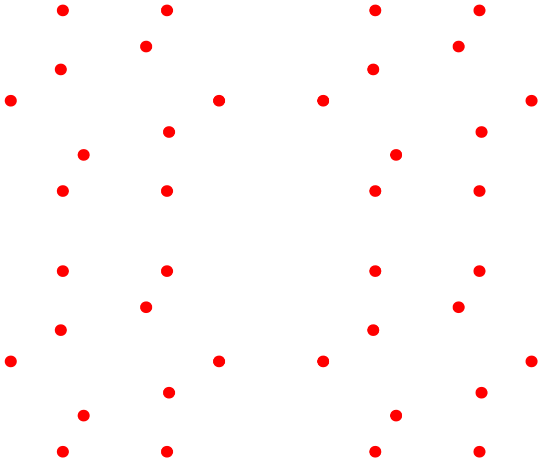
1. What are you talking about?
2. Key questions
3. The new census

Graphs

Quick recap

A graph Γ is a pair $\Gamma = (V, E)$.

Graphs



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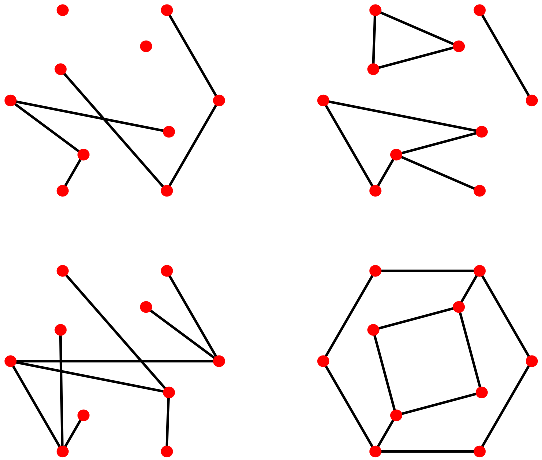
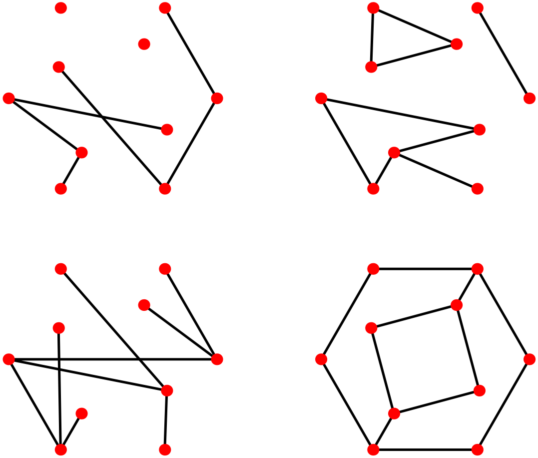


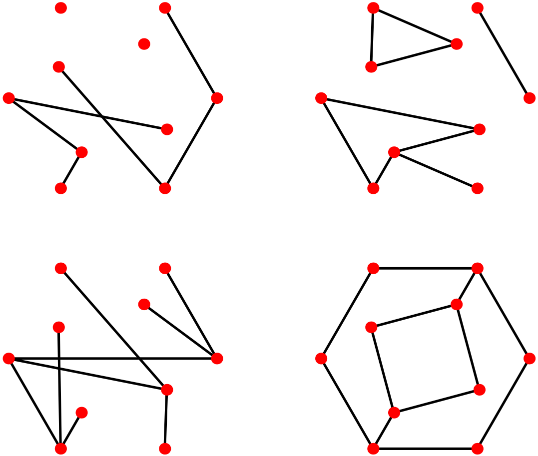
Graphs

Quick recap

A graph Γ is a pair $\Gamma = (V, E)$.
Set V are the vertices or nodes.
Set E are edges.
We deal with simple graphs, so $E \subseteq V \times V$ and edges from v back to v don't exist.



The crucial notion - symmetry



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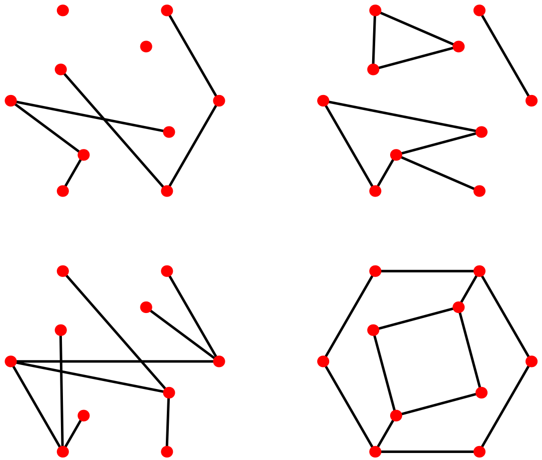

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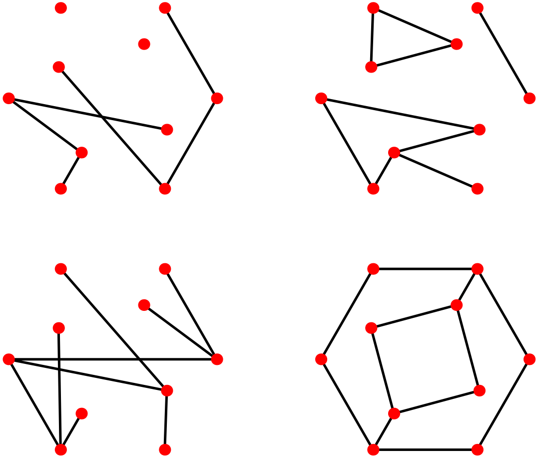

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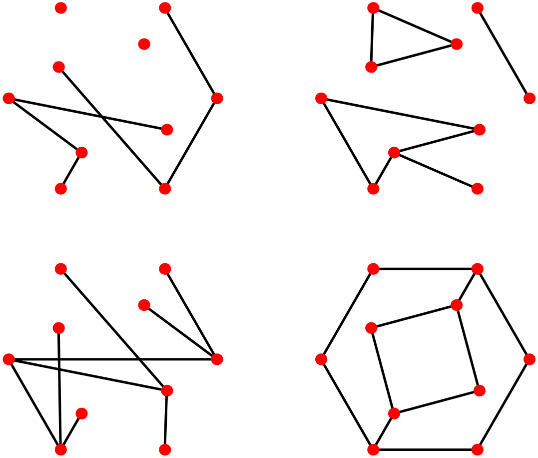

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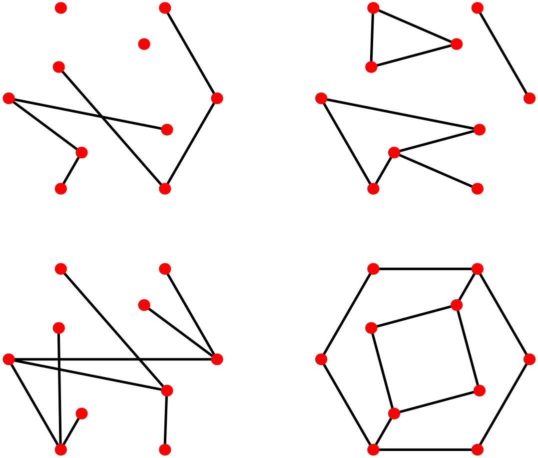
Graph automorphism

Given a graph $\Gamma = (V, E)$, a graph automorphism is a function $\varphi : V \to V$ *(well, actually a permutation)* on vertices such that the (non)-adjacency is preserved:

$$\{u, v\} \in E \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E$$



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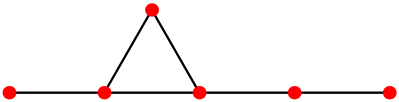

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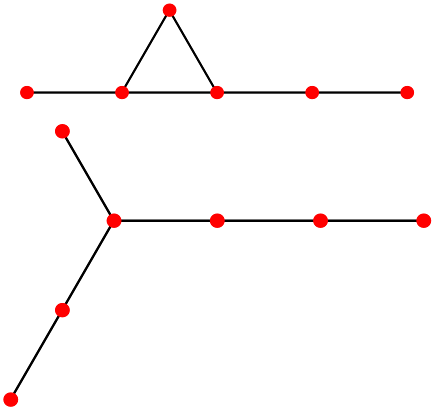

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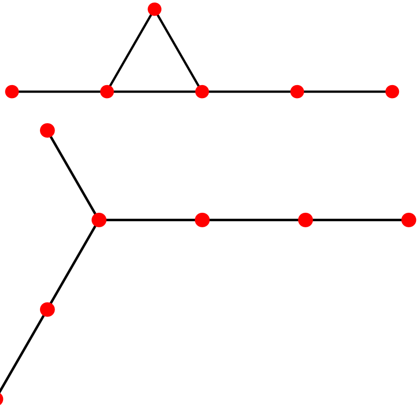

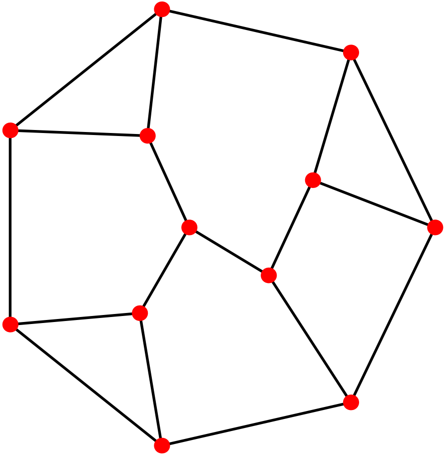

Graphs that are “nice”

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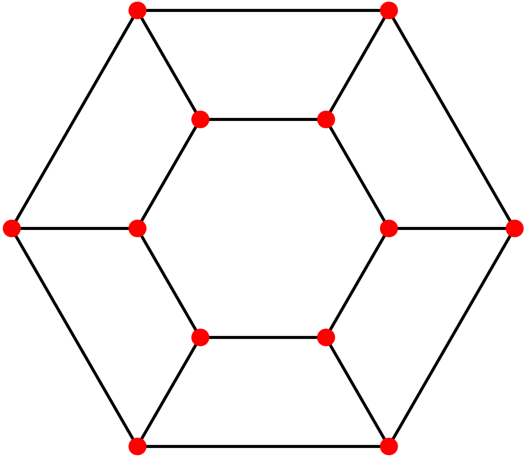
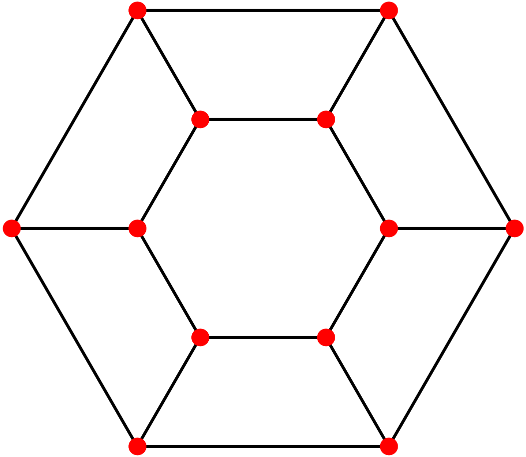


Graphs that are "nice" – vertex-transitive graphs



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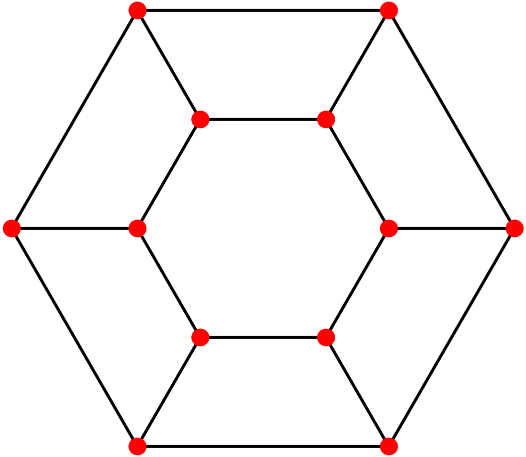

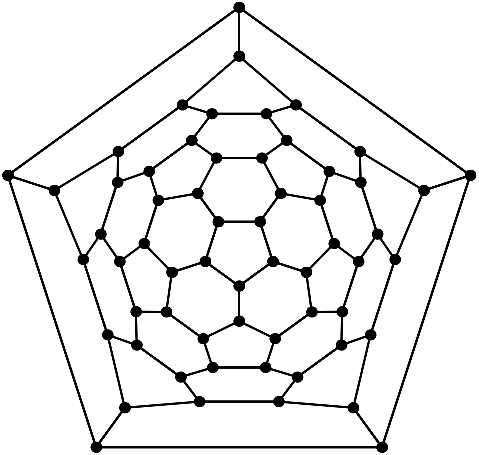

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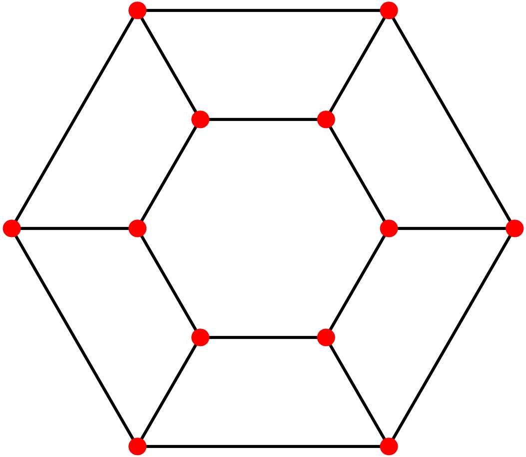

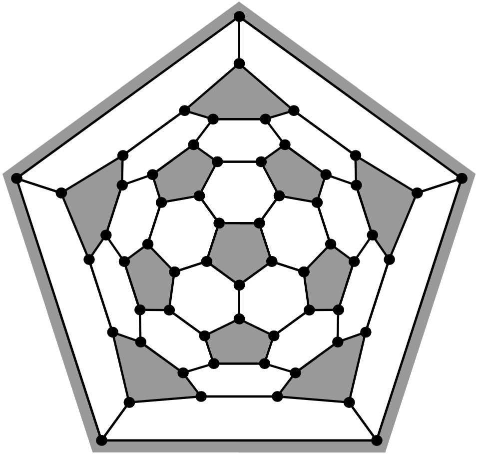

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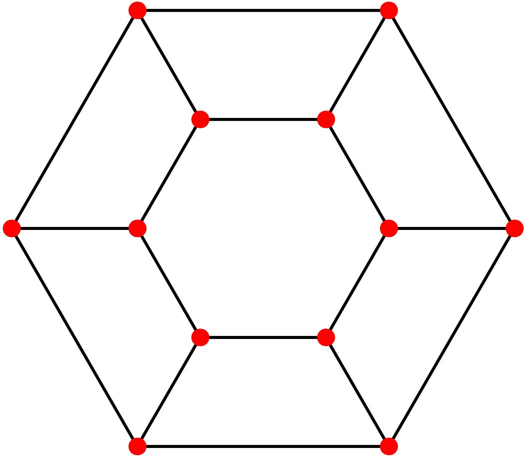

A particular subclass of vertex-transitive graphs

"Cayley graphs are beautiful."

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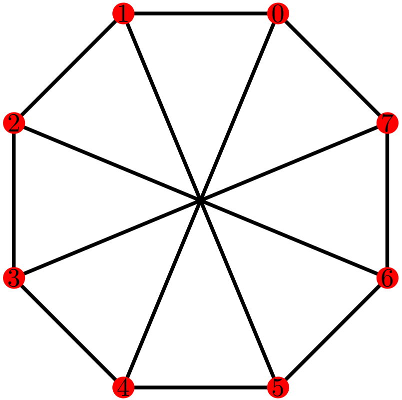

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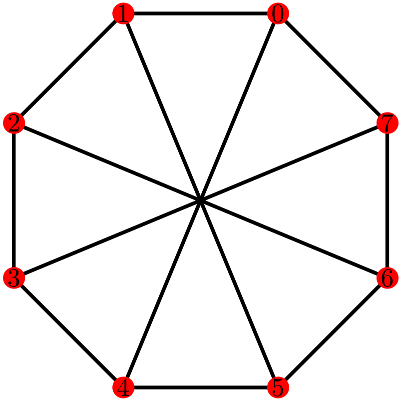

Cayley graph definition

For a given group G and a set $S \subseteq G$, the Cayley Graph $\Gamma = Cay(G, S)$ has

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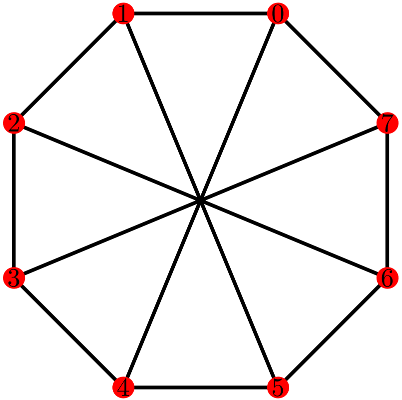

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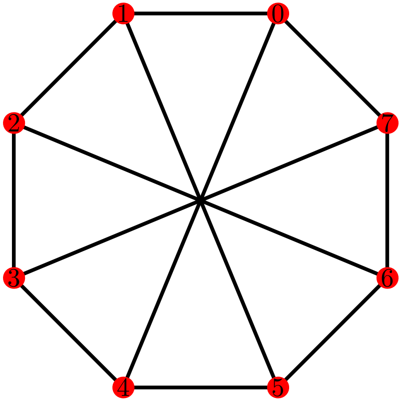

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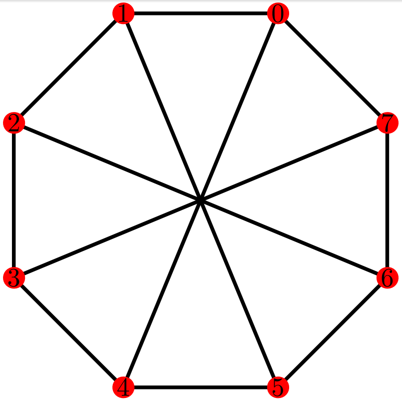


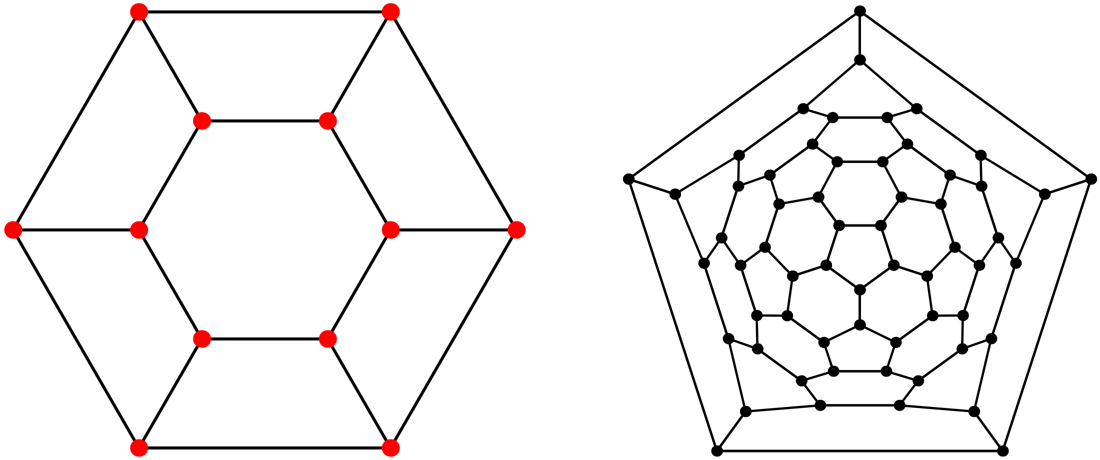
Cayley graph of C_8 with $S = \{1, 4, 7\}$

Cayley graph definition

For a given group G and a set* $S \subseteq G$, the Cayley Graph $\Gamma = Cay(G, S)$ has
$V(\Gamma) = G$, and
$gh \in E(\Gamma) \Leftrightarrow \exists s \in S : gs = h$.

*The set S must not contain e_G and it must be closed under group inverses, i.e. $S = S^{-1}$.

A **proper** subclass



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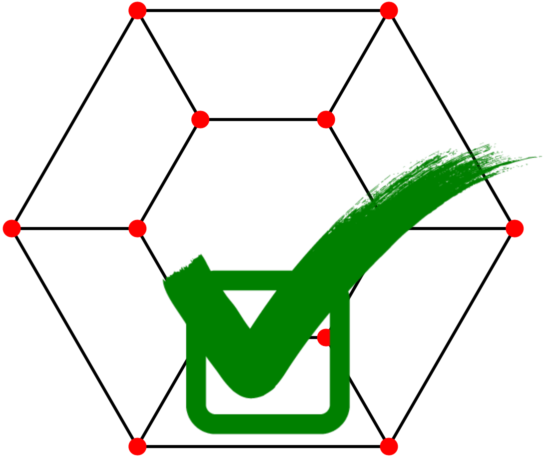

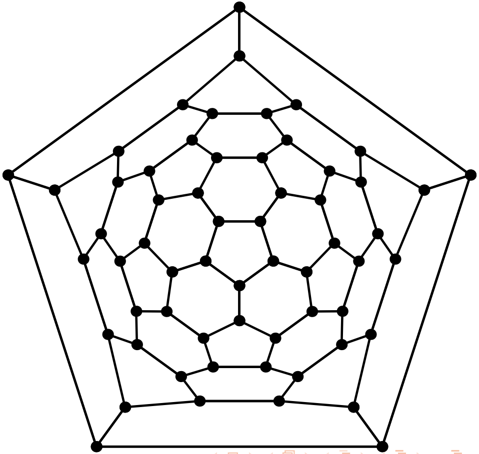

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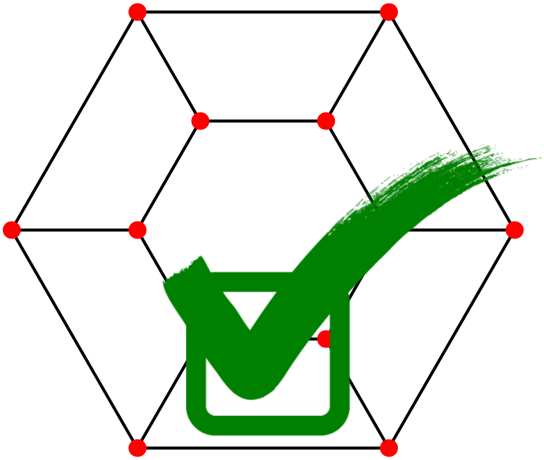

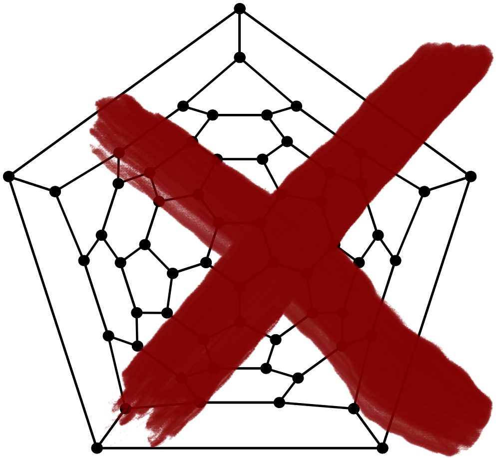

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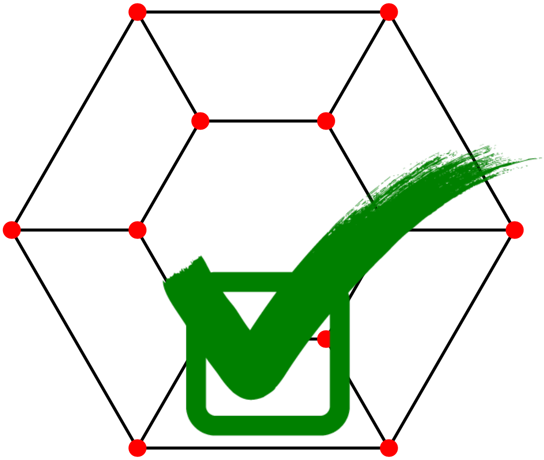
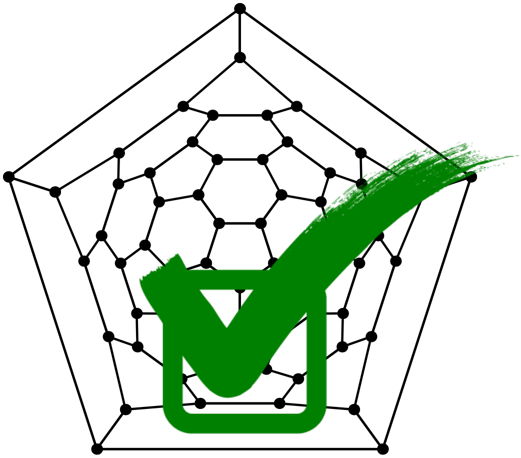
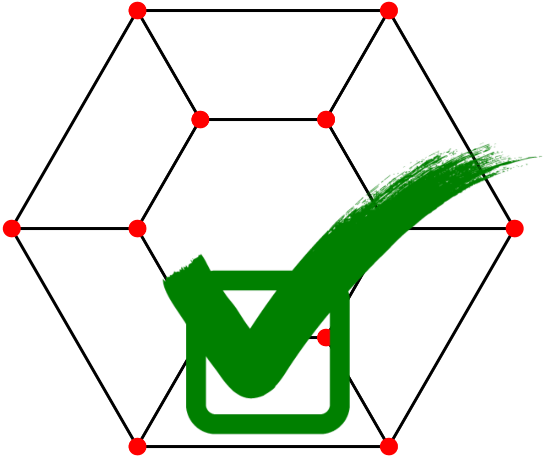
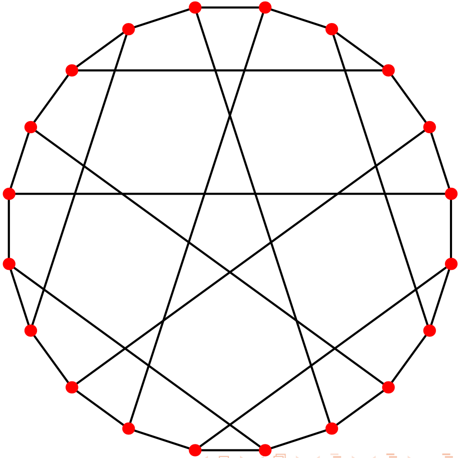




A **proper** subclass - for real this time!





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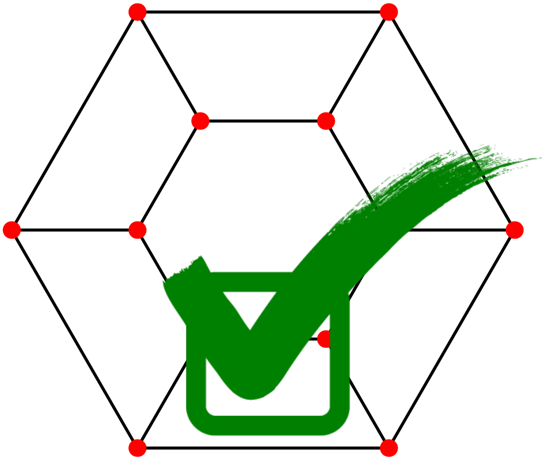
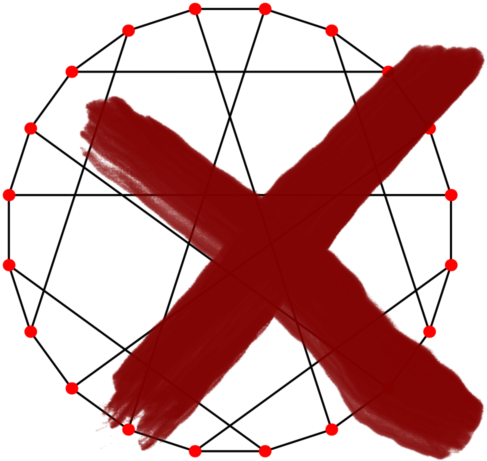
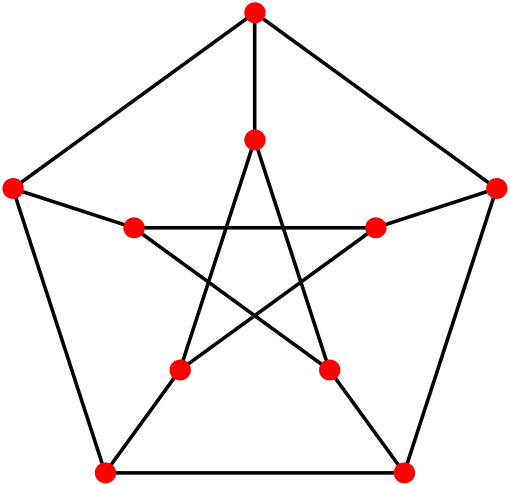
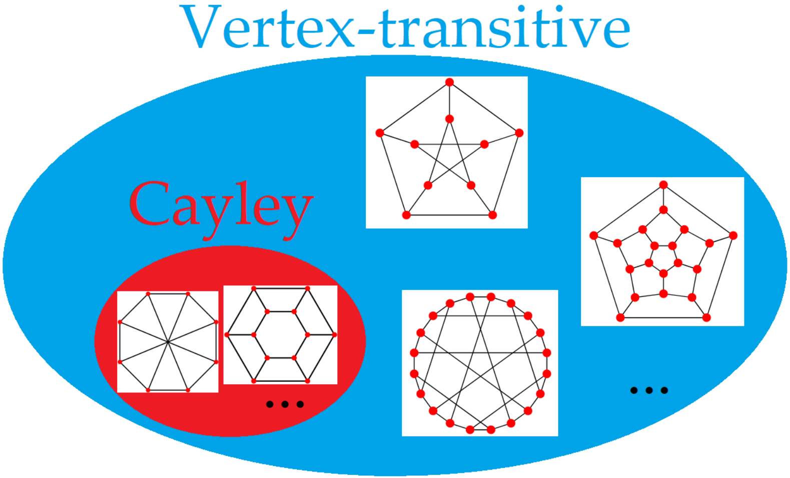




The most famous “non-Cayley” example



The big picture



So... We are done...

… but we can do better!

Which graphs are Cayley?

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ON A CLASS OF FIXED-POINT-FREE GRAPHS[1]

GERT SABIDUSSI

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Theorem *[Sabidussi]*

Graph Γ is Cayley if and only if there exists a subgroup $H \leq Aut(\Gamma)$ acting regularly on the graph's vertices.

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Group acts regularly if each vertex is mapped to each "position" precisely once.

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Group acts <u>regularly</u> if each vertex is mapped to each "position" precisely once.

Because of what will follow, we will call such a group H to be 1-regular.

More precisely, H will act 1-regularly on the graph's vertices.

Which graphs are “almost” Cayley?

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On quasi-Cayley graphs

Ginette Gauyacq *

LaBRI, Université Bordeaux I, U.A. 1304 CNRS, 351, cours de la Libération, F-33405 Talence Cedex, France

Received 1 July 1993; revised 6 February 1996

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Theorem *[Gauyacq]*

Graph Γ is **quasi**-Cayley if and only if there exists a subset $\mathcal{F} \leq Aut(\Gamma)$ acting regularly on the graph’s vertices.

Which graphs are “almost” Cayley? [generalised]

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r-REGULAR FAMILIES OF GRAPH AUTOMORPHISMS

ROBERT JAJCAY AND GARETH A. JONES

Which graphs are “almost” Cayley? [generalised]

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Given a graph Γ, what is the smallest $r > 0$, such that there is a subgroup $G \leq Aut(\Gamma)$ acting r-regularly on Γ's vertices?

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Cayley deficiency $d(\Gamma)$

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asymmetric			

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Initial observations

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Cayley	...		

Initial observations

Γ	$\|V(\Gamma)\|$	$d(\Gamma)$	$r(\Gamma)$
asymmetric	$\dots$	-	-
not vertex-transitive	$\dots$	-	-
Cayley	$\dots$	1	

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not vertex-transitive	...	-	-
Cayley	...	1	1

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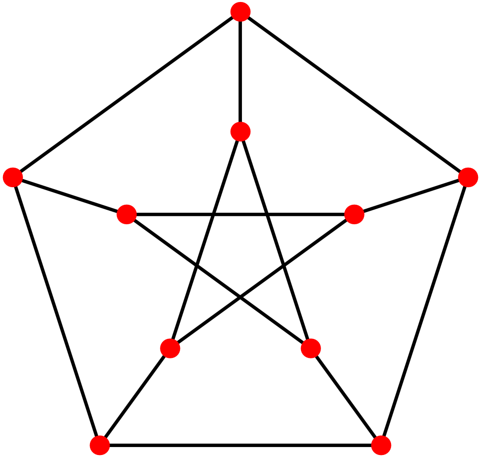
asymmetric	...	-	-
not vertex-transitive	...	-	-
Cayley	...	1	1
Petersen	10		



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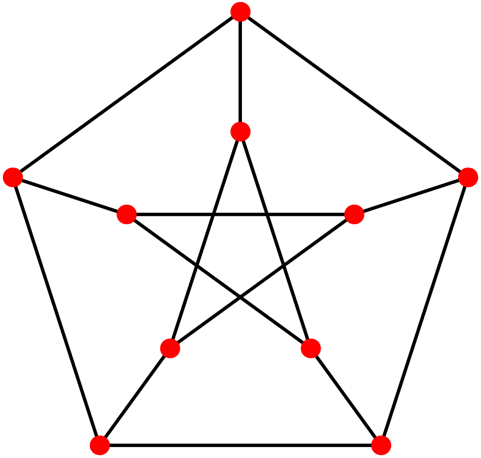
Cayley	...	1	1
Petersen	10	2	



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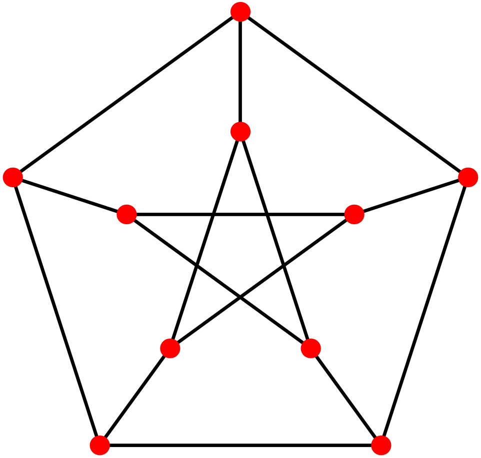

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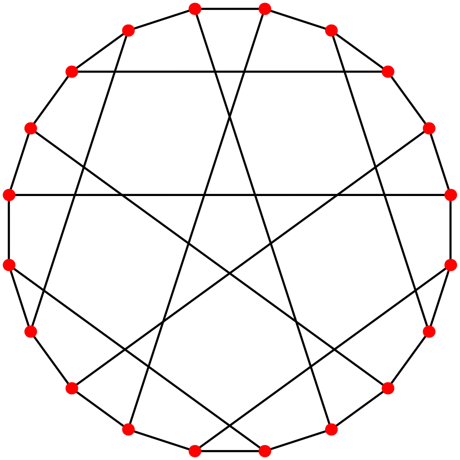
Cayley	...	1	1
Petersen	10	2	1
Desargues	20		



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Petersen	10	2	1
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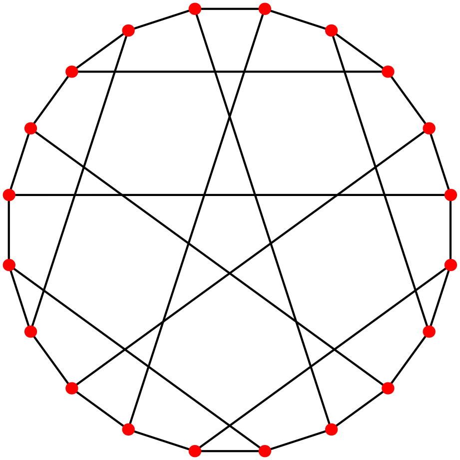

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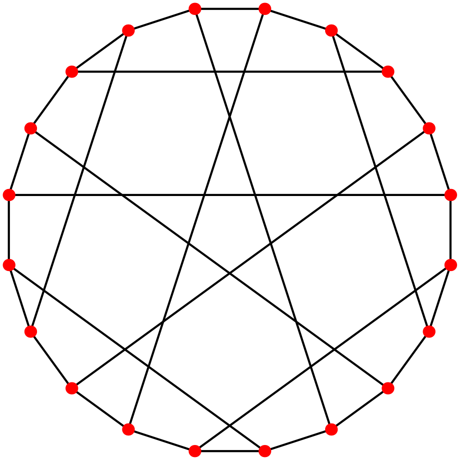
Desargues	20	2	1



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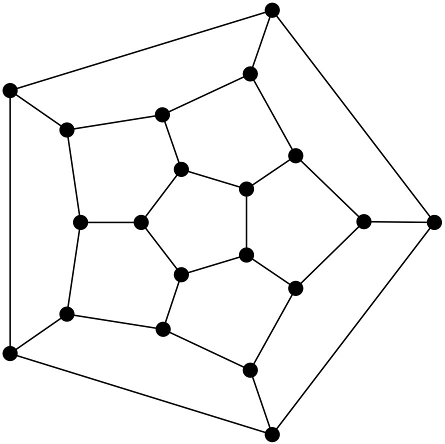
Petersen	10	2	1
Desargues	20	2	1
Dodecahedron	20		



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Desargues	20	2	1
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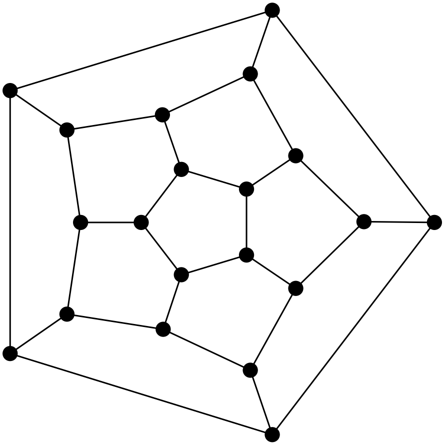

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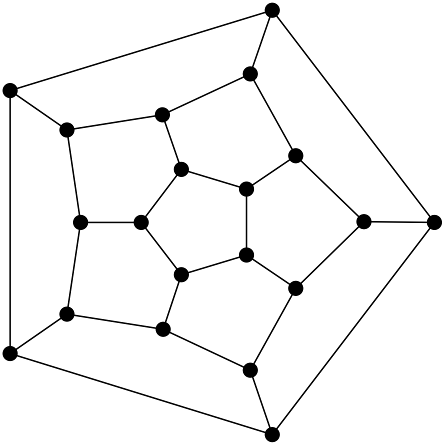

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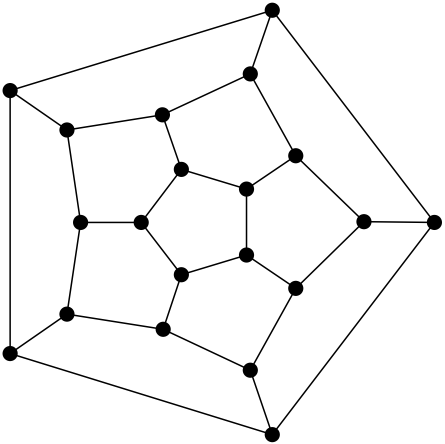
Dodecahedron	20	3	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$



Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10				

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22			

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4		

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	1

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	1
11	8			

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	1
11	8	1		

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	1
11	8	1	3	0

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	1
11	8	1	3	0
12	74			

Focusing on quasi-Cayley deficiency

n	#all VT	#disconnected	#Cayley	$QC(1)$
10	22	4	6	1
11	8	1	3	0
12	74	11	26	0
13	14	1	6	0
14	56	5	23	0

Aside: complements and $d < n/2$

How do we know we are not missing crucial information?

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$$Aut\,(\Gamma) \cong Aut\left(\Gamma^{C}\right)$$

Aside: complements and $d < n/2$

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$$Aut(\Gamma) \cong Aut\left(\Gamma^C\right)$$

Graphs whose degree is the same as their complement?

- If the common degree is d, we must have $2d + 1$ vertices.

Aside: complements and $d < n/2$

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Graphs whose degree is the same as their complement?

- If the common degree is d, we must have $2d + 1$ vertices.
- Such a graph cannot be disconnected, as there are not enough vertices.

Aside: complements and $d < n/2$

How do we know we are not missing crucial information?

$$Aut(\Gamma) \cong Aut\left(\Gamma^C\right)$$

Graphs whose degree is the same as their complement?

- If the common degree is d, we must have $2d + 1$ vertices.
- Such a graph cannot be disconnected, as there are not enough vertices.
- Thus the graphs can never be characterised differently.

At $n = 15$ something interesting happens!

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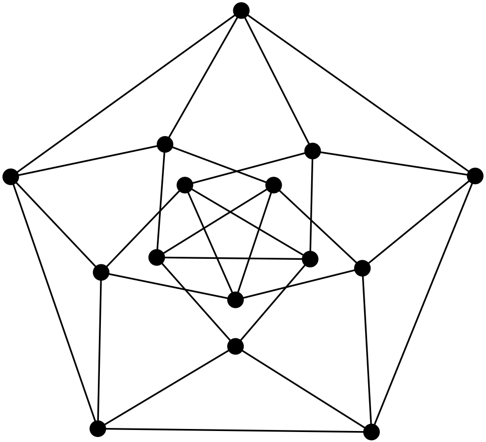

Aside: Wait what? Line graphs?

$$QC(\text{Petersen}) = 1; \; QC(LG(\text{Petersen})) = 2$$

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$$QC(LG^2(\text{Petersen})) = 2;\ QC(LG^3(\text{Petersen})) = \text{-}$$

Obstruction

In order for $LG(\Gamma)$ to be interesting to us

Aside: Wait what? Line graphs?

$$QC(\text{Petersen}) = 1; \; QC(LG(\text{Petersen})) = 2$$

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In order for $LG(\Gamma)$ to be interesting to us,
Γ must be not only be vertex-transitive

Aside: Wait what? Line graphs?

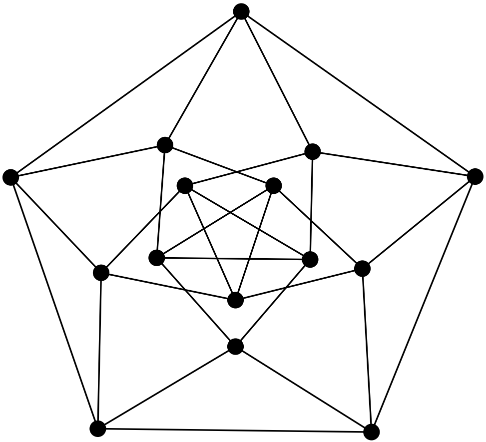
$$QC(\text{Petersen}) = 1;\ QC(LG(\text{Petersen})) = 2$$

$$QC(LG^2(\text{Petersen})) = 2;\ QC(LG^3(\text{Petersen})) = -$$

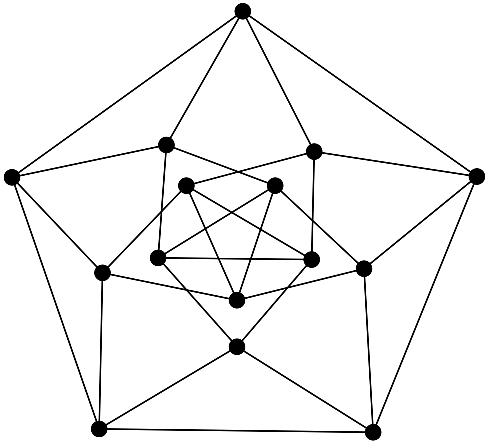
Obstruction

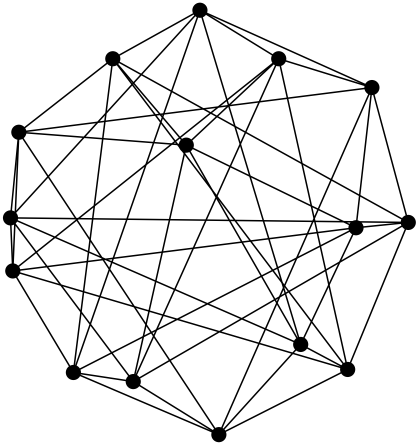
In order for $LG(\Gamma)$ to be interesting to us,
Γ must be not only be vertex-transitive,
but also **edge-transitive**.

At $n = 15$ something interesting happens!



At $n = 15$ something interesting happens!





Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48				

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18		

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18	0	

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18	0	2

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18	0	2
16	286	14	125	4	0
17	36	1	17	0	0
18	380	15	173	2	0
19	60	1	29	0	0

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18	0	2
16	286	14	125	4	0
17	36	1	17	0	0
18	380	15	173	2	0
19	60	1	29	0	0
20	1214				

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18	0	2
16	286	14	125	4	0
17	36	1	17	0	0
18	380	15	173	2	0
19	60	1	29	0	0
20	1214	24	544		

Back with the table

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$
10	22	4	6	1	0
11	8	1	3	0	0
12	74	11	26	0	0
13	14	1	6	0	0
14	56	5	23	0	0
15	48	4	18	0	2
16	286	14	125	4	0
17	36	1	17	0	0
18	380	15	173	2	0
19	60	1	29	0	0
20	1214	24	544	33*	0

$n = 20$ brings in more record graphs!

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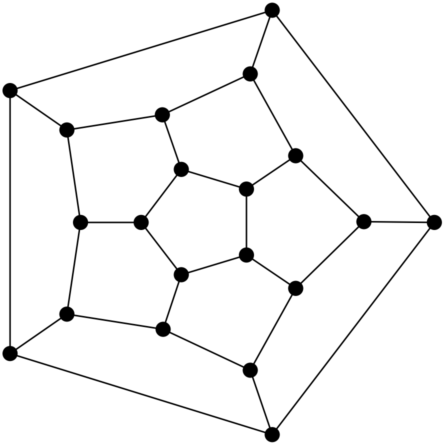
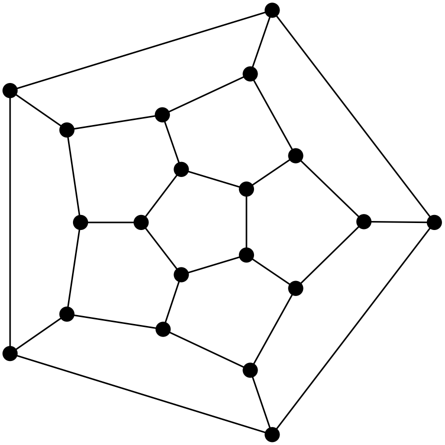
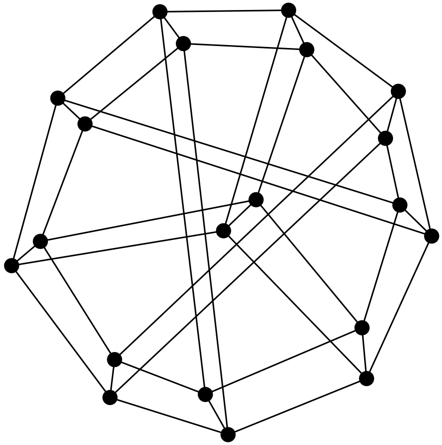


$n = 20$ brings in more record graphs!





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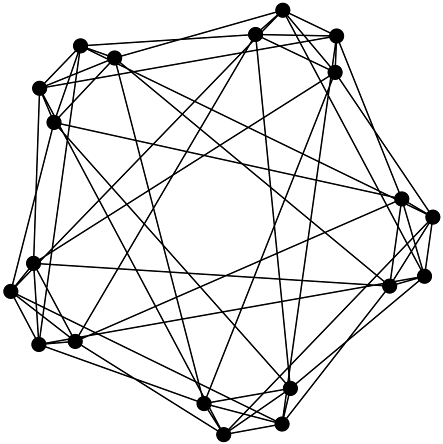

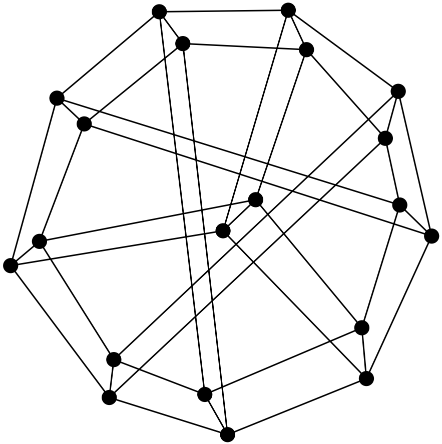

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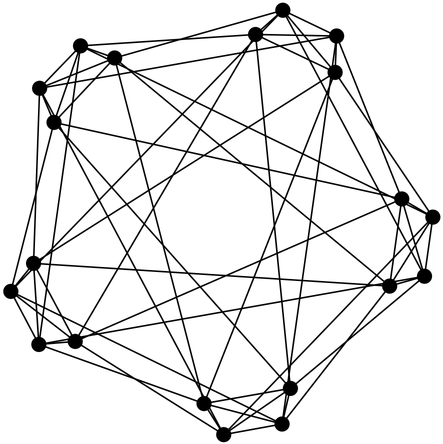

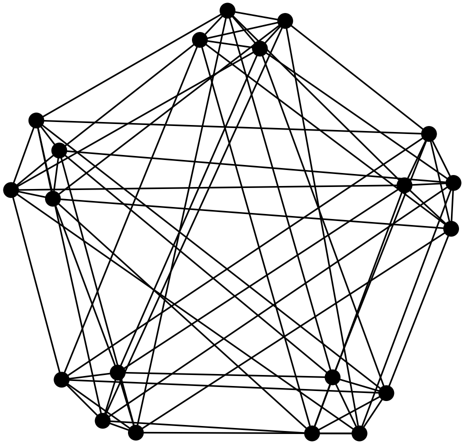

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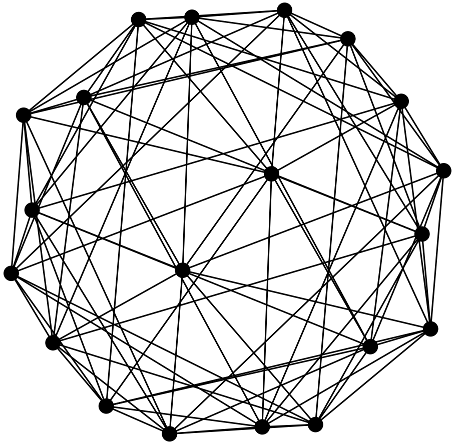

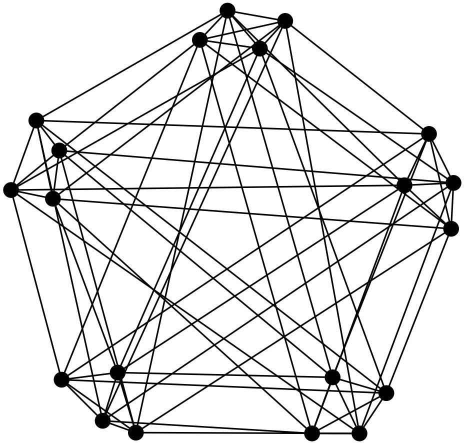

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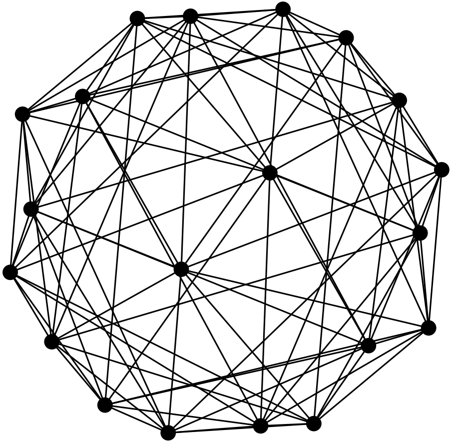
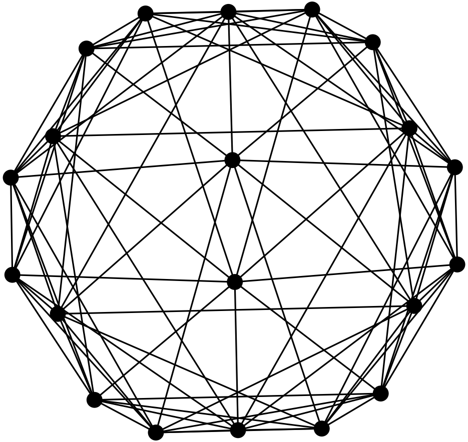
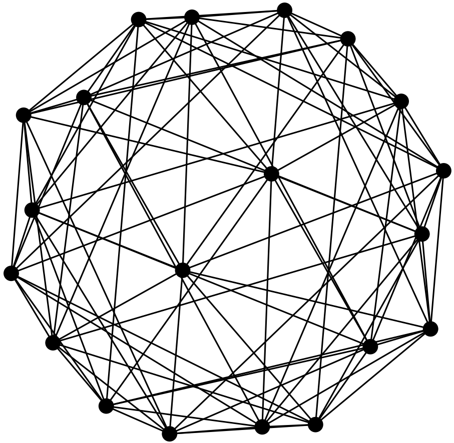
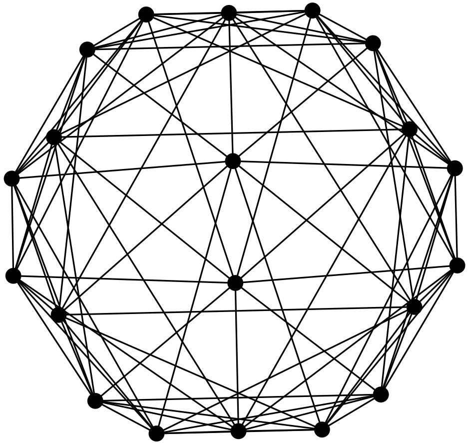




$n = 20$ brings in more record graphs! *House of Graphs "MQC"*





Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
10	22	4	6	1	0	0
11	8	1	3	0	0	0
12	74	11	26	0	0	0
13	14	1	6	0	0	0
14	56	5	23	0	0	0
15	48	4	18	0	2	0
16	286	14	125	4	0	0
17	36	1	17	0	0	0
18	380	15	173	2	0	0
19	60	1	29	0	0	0
20	1214	24	544	33*	0	6

Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
20	1214	24	544	33^*	0	6

Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
20	1214	24	544	33*	0	6
21	240	5	115	0	0	0
22	816	9	399	0	0	0
23	188	1	93	0	0	0

Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
20	1214	24	544	33*	0	6
21	240	5	115	0	0	0
22	816	9	399	0	0	0
23	188	1	93	0	0	0
24	15506					

Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
20	1214	24	544	33*	0	6
21	240	5	115	0	0	0
22	816	9	399	0	0	0
23	188	1	93	0	0	0
24	15506	84	7613	56	0	0

Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
20	1214	24	544	33*	0	6
21	240	5	115	0	0	0
22	816	9	399	0	0	0
23	188	1	93	0	0	0
24	15506	84	7613	56	0	0
25	464	3	229	0	0	0
26	4236	15	2037	66	0	0
27	1434	9	708	0	0	0

Guess who's back... Back again.

n	#all VT	#disconnected	#Cayley	$QC(1)$	$QC(2)$	$QC(3)$
20	1214	24	544	33*	0	6
21	240	5	115	0	0	0
22	816	9	399	0	0	0
23	188	1	93	0	0	0
24	15506	84	7613	56	0	0
25	464	3	229	0	0	0
26	4236	15	2037	66	0	0
27	1434	9	708	0	0	0
28	25850	?	?	?	?	?
29	1182	1	590	0	0	0
30	46308	?	?	?	?	?

Everyone faces some mountains

That is all I have prepared

Thank you for your attention!