Symmetry-preserving operations and the genus of graphs

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May 4, 2026

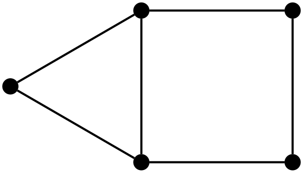


Figure: A simple graph

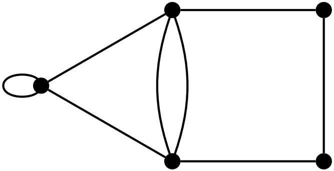


Figure: A (multi)graph

Definition

An embedding is a drawing of a (multi)graph on a surface, where the edges do not intersect.

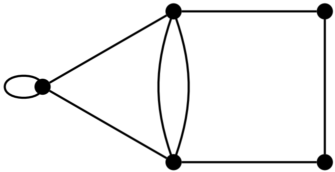
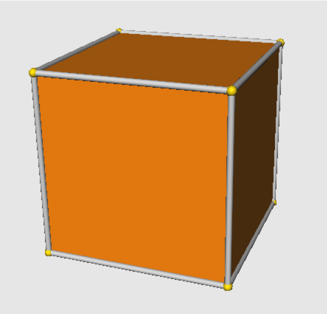
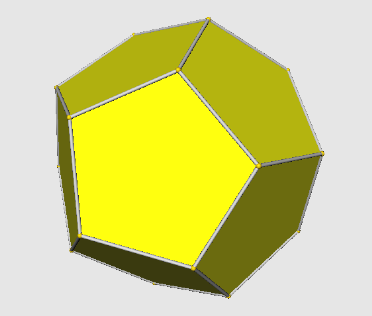


Figure: An embedding in the plane

Polyhedra are also embeddings.



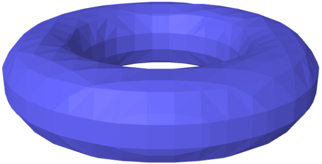
(a) Cube



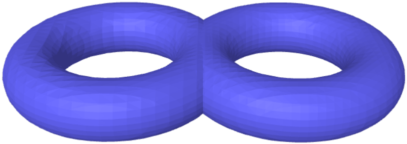
(b) Dodecahedron

Introduction

Graphs can also be embedded on surfaces of higher genus.



(a) Torus: genus 1



(b) Double torus: genus 2

The genus of a surface is the number of holes in the surface.

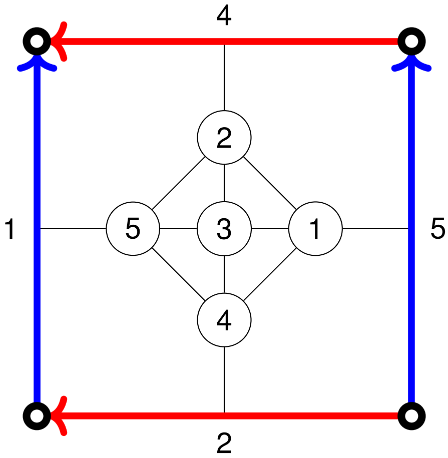


Figure: Embedding of K5 on a torus

Definition

The genus of an embedding is the genus of the surface it's embedded in.
The genus of a graph is the minimal possible genus of an embedding of that graph.

'Local' operations that preserve symmetry

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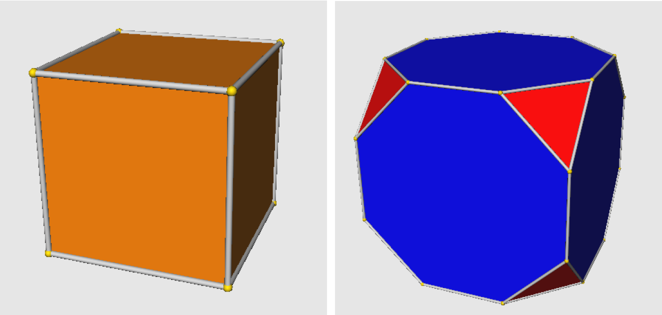


Figure: Truncating a cube

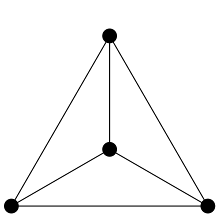
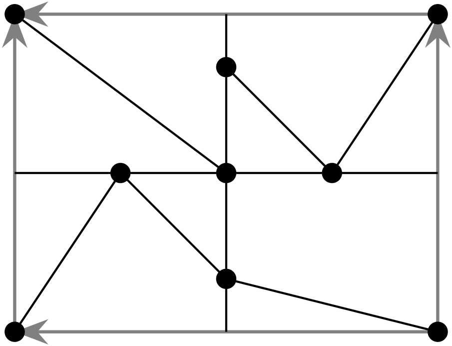
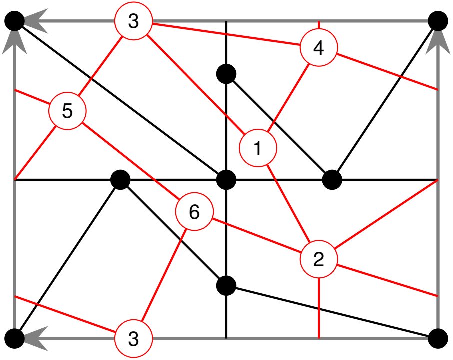
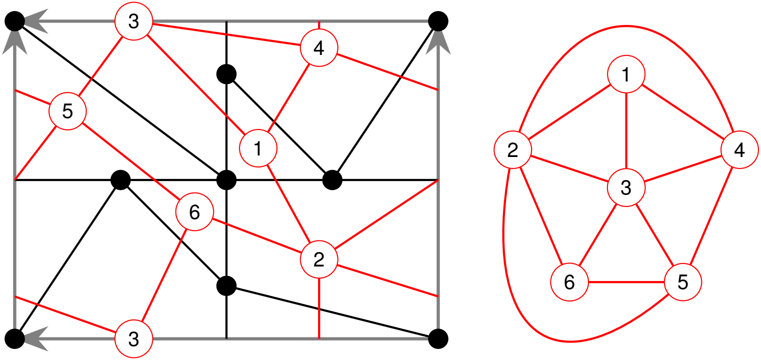


Figure: Truncation of an embedding of K4 in the plane.




3
4
5
1
6
2
3


3
4
5
1
6
2
3
1
2
4
3
6
5

By connecting 2 triangles, two embeddings M_1 and M_2 can be combined to create an embedding $M_1 \triangle M_2$ such that

- $mg(M_1) + mg(M_2) = mg(M_1 \triangle M_2)$,
- $mg(Dual(M_1)) + mg(Dual(M_2)) = mg(Dual(M_1 \triangle M_2))$.

If M is an embedding for which Dual reduces the genus from 1 to 0, then $M \triangle \ldots \triangle M$ is an embedding on which Dual reduces the genus from n to 0.

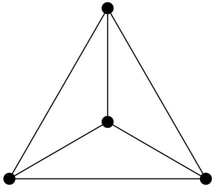


Figure: Join applied to a tetrahedron.

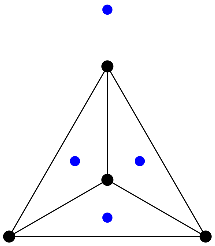
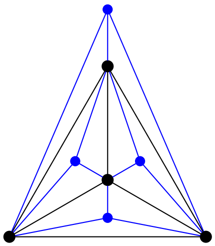


Figure: Join applied to a tetrahedron.



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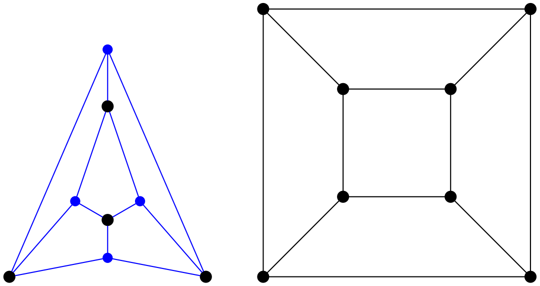


Figure: Join applied to a tetrahedron.

Theorem

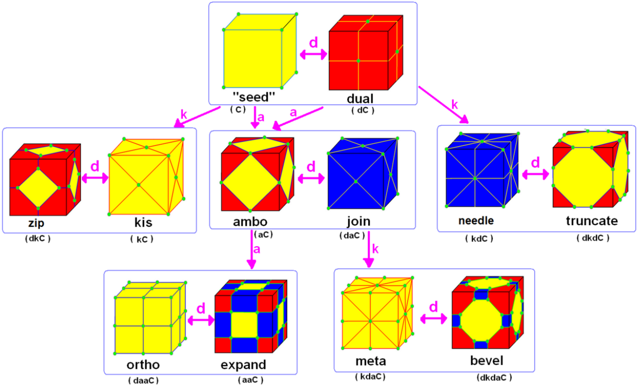
The result of applying Join is always a minimum genus embedding as long as it is simple.

Proof.

Euler's formula: $2g = E - V - F + 2$ hence to decrease the genus you must create more faces. So the *average face size* has to decrease.
The result of Join is an embedding of a bipartite graph, and all its faces have size 4.
The faces of a bipartite embedding are at least size 4, so the average face size cannot go below 4. □

Lsp-operations are a large class of symmetry-preserving operations.

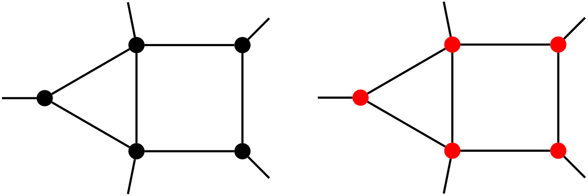
- Uniform definition.
- Allow us to state general facts about many operations.
- Contains all of the operations that have historically been studied.



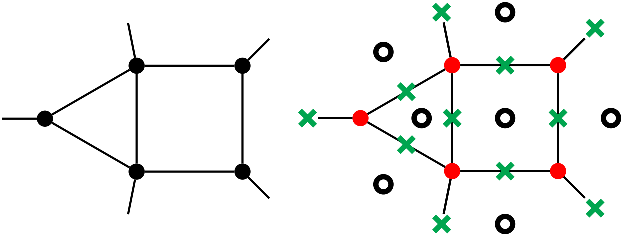
Twelve forms created from three operations: dual, ambo, kis

Conway operations $\subset$ lsp-operations.

We need the barycentric subdivision to define lopsp-operations



We need the barycentric subdivision to define lopsp-operations



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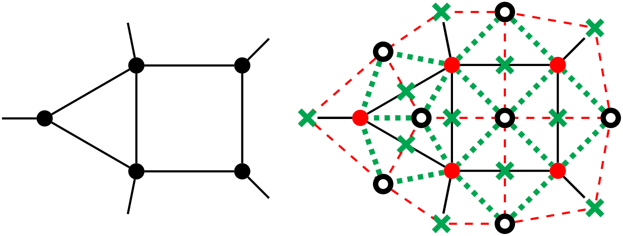
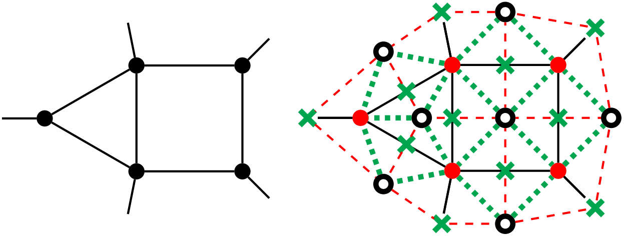
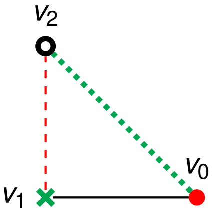
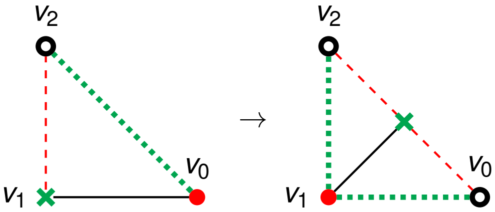


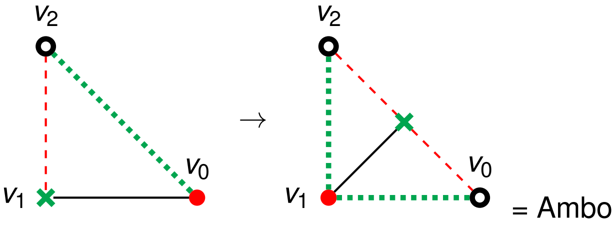
We need the barycentric subdivision to define lopsp-operations



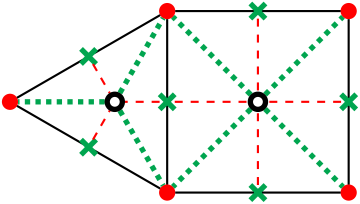
Colouring: $0 =$ ●, $1 =$ ✖, $2 =$ ○.


v_2
v_0
v_1

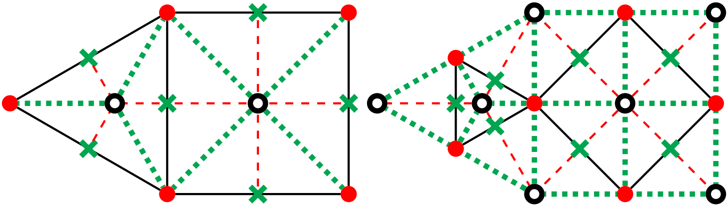

v_2
v_1
v_0
→
v_2
v_1
v_0


v2
v1
v0
→
v2
v1
v0
= Ambo

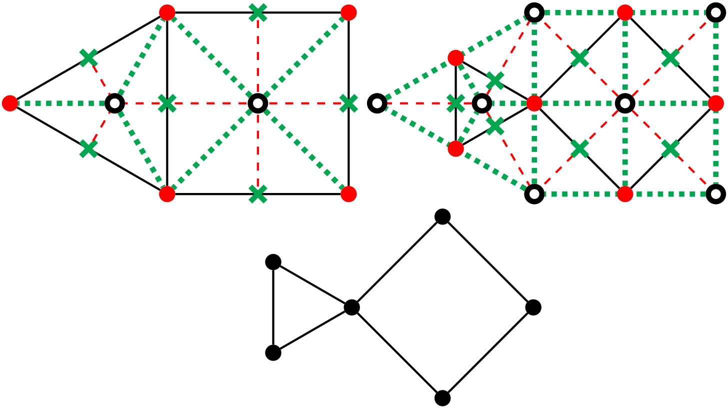
Lsp-operations



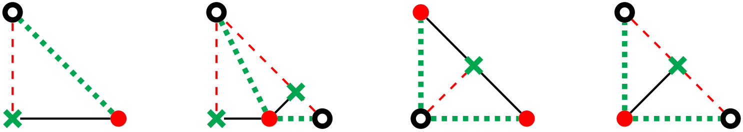
Lsp-operations



Lsp-operations



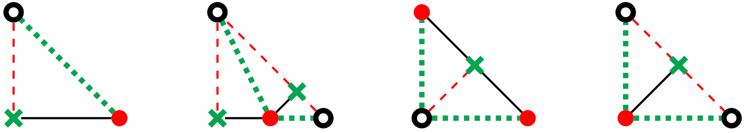
Lsp-operations



(a) Id (b) Truncation (c) Join (d) Ambo

Figure: Some lsp-operations.

Lsp-operations



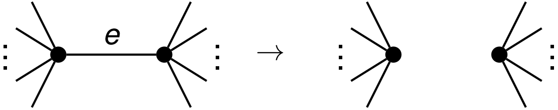
(a) Id (b) Truncation (c) Join (d) Ambo

Figure: Some lsp-operations.

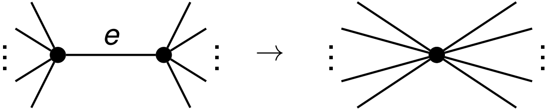
Requirements for a valid chamber patch:

- locally looks like a barycentric subdivision.
- Has the ‘triangle shape’ like a chamber.

G is a minor of H if G can be gotten from H by deleting or contracting edges. $g(G) \leq g(H)$ if G is a minor of H.



(a) d_e



(b) c_e

The minor relation is naturally extended to lsp-operations.
We proved that all lsp-operations either have Id or Join as a minor, or it decreases the genus, on some polyhedral embeddings.
The operations that can decrease the genus have a structure similar to Ambo.

Theorem

Let O be a lopsp-operation, the following are equivalent.

1. *For every polyhedral minimum genus embedding M,* $mg(M) = mg(O(M))$.
2. *The colour of* v_0 *in O is* 0*, or* v_0 *is shrinkable.*

Overview and further research

- We know which operations can decrease the genus on polyhedral embeddings. Can it be extended to larger classes of embeddings?
- We have polyhedral embeddings on which these operations decrease the genus from $3n$ to $2n$. Can this ratio go lower?
- On polyhedral embeddings, the genus cannot be reduced to 0. This is the only lower bound we have so far.