Searching for hypergraphs with prescribed regular automorphism groups of small orders

Robert Jajcay
Comenius University
joint work with Isabel Hubard and Tatiana Jajcayova

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Regular Group Actions

Definition

Let G be a group acting on a set V.

- The action of G on V is said to be **transitive** if for any pair of elements $u, v \in V$ there exists an element $g \in G$ such that $g \cdot u = v$.
- The action of G on V is said to be **regular** if for any pair of elements $u, v \in V$ there exists *exactly one* element $g \in G$ such that $g \cdot u = v$.

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Equivalently, an action of G on V is regular iff

- G acts transitively on V and $Stab_G(v) = 1_G$, for all $v \in V$
- G acts transitively on V and $|G| = |V|$
- G acts transitively on V and the only element of G that fixes an element of V is 1_G

Regular Representations of Groups

Theorem (Cayley)

Every group G acts regularly on itself via (left) multiplications, i.e., G is isomorphic to the group $G_L = \{\sigma_g \mid g \in G\}$ of (left) translations:

$$\sigma_g(h) = g \cdot h, \quad \text{for all } h \in H$$

Note:

- The action of G_L on G is regular.
- Every regular action of G on a set V can be viewed as the action of G_L on G.

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  - One may be asking *which groups are* **isomorphic** *to the (full) automorphism groups of structures from the class* or *which permutation representations of groups are* **equal** *to the (full) automorphism groups of structures from the class*

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- Each finite group (in its regular representation G_L) is equal to a group of automorphisms of any corresponding Cayley graph (but not necessarily the full group of automorphisms)
- Finite groups for which there exists a Cayley graph whose *full automorphism group is equal to the group of permutations of their left regular representation* G_L have been fully classified after a considerable effort; such representations are called **G**raphical **R**egular **R**epresentations

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Theorem (RJ)

A finite group G can be represented as a regular full automorphism group of some some hypergraph if and only if G is not one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ *or* $\mathbb{Z}_2^2$.

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We address this same question for k-uniform hypergraphs:
Given a (finite) group G, find the set of all k's, $1 \leq k \leq |G| - 1$, for which G can be regularly represented as the full automorphism group of a k-regular hypergraph,

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Given a (finite) group G, find the set of all k's, $1 \leq k \leq |G| - 1$, for which G can be regularly represented as the full automorphism group of a k-regular hypergraph, i.e., determine the set of k's for which there exists a set of hyperedges $\mathcal{H} \subseteq \mathcal{P}_k(G)$ such that the combinatorial structure $(G, \mathcal{H})$ satisfies $Aut(G, \mathcal{H}) = G_L$.

Lemma (RJ)

Let $\mathcal{H} = (V, \mathcal{B})$ *be a hypergraph, and let* k *be a non-negative integer* $0 \leq k \leq |V|$. *Then*

(i) $Aut(V, \mathcal{B}) = Aut(V, (\mathcal{B} - \mathcal{P}_k(V)) \cup (\mathcal{P}_k(V) - \mathcal{B}))$,

(ii) $Aut(V, \mathcal{B}) = Aut(V, (\mathcal{B} - \mathcal{P}_k(V)) \cup \{B^c \mid B \in \mathcal{B} \cap \mathcal{P}_k(V)\})$,

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Corollary

The set of k's admitting a k-HRR for a finite group of order n is symmetric; k admits an HRR if and only if $n - k$ does.

Regular Representation of $\mathbb{Z}_4$

- $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$

$$\sigma_0 = (0)(1)(2)(3), \quad \sigma_1 = (0,1,2,3),$$
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- $k \neq 2$, based on the classification of GRR's **or** based on the observation that the set of edges of a graph whose automorphism group is $(\mathbb{Z}_4)_L$ must consist of a union of orbits of $(\mathbb{Z}_4)_L$ on $\mathcal{P}_2(\{0, 1, 2, 3\})$:
$$\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 0\}$$
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... and after all, the theorem from two slides back stated that $\mathbb{Z}_4$ **cannot be represented via any hypergraph**

Regular Representations of $\mathbb{Z}_n$'s on k-Hypergraphs

Theorem (Hubard, RJ, Jajcayova)

A cyclic group $\mathbb{Z}_n$ *can be regularly represented on a k-hypergraph if and only if* $n \neq 3, 4, 5$, *and if* $n \geq 6$, *the spectrum of k's admitting a k-uniform* **H***ypergraphical* **R***egular* **R***epresentation k-HRR for* $\mathbb{Z}_n$ *is* $3, \ldots, n-3$.

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Proof.

- $\mathcal{B} = \{ \{i, i+1, \ldots, i+k-2, i+k-1\} \mid 0 \leq i \leq n-1 \}$
  $\cup \ \{ \{i, i+1, \ldots, i+k-2, i+k\} \mid 0 \leq i \leq n-1 \}$

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- $Aut(\mathbb{Z}_n, \mathcal{B}) \leq Aut(\mathbb{Z}_n, \{i, i+1\})$
- $Stab(0) = 1$
- cyclic groups of order ≥ 6 do not admit graphical regular representations

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This means that there is **no infinite family of groups not admitting a** 3-**HRR** and the classification of finite groups admitting a 3-HRR is (at least hypothetically) computationally achievable

Constructing k-HRR's for specific finite G

A k-regular hypergraph that admits a regular group of automorphisms G_L is isomorphic to $(G, \mathcal{H})$, where $\mathcal{H}$ consists of *a union of orbits* $\cup_{i=1}^{s} G \cdot H_i$, with H_i being an arbitrary representative of the i-th orbit of the left multiplication action of G on the set $\mathcal{P}_k(G)$

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We can choose the orbit representatives to be k-subests of G containing 1_G:

$$\mathcal{H} = \cup_{i=1}^{s} G \cdot \{1_G, h_{2,i}, h_{3,i}, \dots, h_{k,i}\} = \cup_{i=1}^{s} \cup_{g \in G} g \cdot \{1_G, h_{2,i}, h_{3,i}, \dots, h_{k,i}\}.$$

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A finite group G admits a k-HRR if and only if there exists a union $\mathcal{H}$ of orbits of G_L on $\mathcal{P}_k(G)$ such that $Aut(G, \mathcal{H}) = G_L$.

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A finite group G admits a k-HRR if and only if there exists a union $\mathcal{H}$ of orbits of G_L on $\mathcal{P}_k(G)$ such that $Aut(G, \mathcal{H}) = G_L$.

In addition, if $(G, \mathcal{H})$ is a k-HRR for G, $(G, \mathcal{H})$ must be connected, and $(G, \mathcal{H})$ is connected if and only if the union $\cup_{i=1}^{s} \{1_G, h_{2,i}, h_{3,i}, \ldots, h_{k,i}\}$ generates G.

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- **However**, for a group G of order n, G_L has *at least* $\binom{n}{3}/n \approx n^2$ orbits in its action of $\mathcal{P}_3(G)$ which yields at least $\approx 2^{n^2}$ possible sets $\mathcal{H}$

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- **Moreover**, this kind of checking would have to be done for all groups of orders ≤ 3300!!!

Computational Results for Groups of Order ≤ 32

Theorem (Mihálová '22)

The spectra of k's, $0 \leq k \leq |G|$, for groups G of orders not exceeding 32 for which there exist k-uniform regular representations are as follows:

(i) $\emptyset$ (i.e., G does not admit a k-uniform regular representation), if G is one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2^2$;

(ii) $\{0,1\}$, if G is trivial;

(iii) $\{0,1,2\}$, if $G = \mathbb{Z}_2$;

(iv) $\{4\}$, if G is one of the groups $\mathbb{Q}_8$ or $\mathbb{Z}_2^3$;

(v) $\{3,\ldots,|G|-3\}$, if G is one of the groups of order ≤ 32 that do not admit a GRR and G is different from $\mathbb{Q}_8$ or $\mathbb{Z}_2^3$;

(vi) $\{2,\ldots,|G|-2\}$, in all other cases.

Theoretical Results

Theorem (Hubard, RJ, Jajcayová '25+)

All finite groups G other than the groups

$$\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_5, \mathbb{Z}_2^3, \text{ and } \mathbb{Q}_8,$$

the abelian groups

$$\mathbb{Z}_5^4, \mathbb{Z}_5^3, \mathbb{Z}_4^4, \mathbb{Z}_4^3 \times \mathbb{Z}_2, \mathbb{Z}_4^2 \times \mathbb{Z}_2^2, \mathbb{Z}_4^3, \text{ and } \mathbb{Z}_3^4,$$

and the generalized dicyclic groups with kernels

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The orders not covered by the theorem are abelian groups of orders $625, 125, 128, 81$ and 64 and generalized dicyclic groups of orders 128 and 64

To complete the classification computationally, we can take advantage of:

Theorem (Hubard, RJ, Jajcayová '25+)

Let G be a finite group, and let $(G, \mathcal{H})$ be a connected k-hypergraph admitting G_L as its automorphism group. Let X be a generating set for G. If $\varphi(x) = x$, for all $x \in X$ and for all φ belonging to the stabilizer of 1_G in $Aut(G, \mathcal{H})$, then $(G, \mathcal{H})$ is a k-HRR for G.

Corollary

Let G be a finite group, and let $(G, \mathcal{H})$ be a connected k-hypergraph admitting G_L as its automorphism group, i.e.,

$$\mathcal{H} = \cup_{i=1}^{s} G \cdot \{1_G, h_{2,i}, h_{3,i}, \ldots, h_{k,i}\}.$$

Let V consist of 1_G, the elements $h_{j,i}^{\pm 1}$, and the elements $h_{j,i}^{-1} h_{j',i}$, $2 \leq j \neq j' \leq k$, $1 \leq i \leq s$, let $\mathcal{H}'$ be the set of hyperedges

$$\begin{aligned} \{1_G, h_{2,i}, h_{3,i}, \ldots, h_{k,i}\}, \\ \{h_{2,i}^{-1}, 1_G, h_{2,i}^{-1} h_{3,i}, \ldots, h_{2,i}^{-1} h_{k,i}\}, \\ \ldots, \\ \{h_{k,i}^{-1}, h_{k,i}^{-1} h_{2,i}, h_{k,i}^{-1} h_{3,i}, \ldots, 1_G\}, \\ 1 \leq i \leq s, \end{aligned}$$

and let Γ be the vertex-hyperedge incidence graph of $(V, \mathcal{H}')$. If the stabilizer subgroup of 1_G in $Aut(\Gamma)$ fixes a set $X \subseteq V$ that generates G, then $(G, \mathcal{H})$ is a k-HRR for G.

To complete the classification computationally:

- Our corollary significantly simplifies the process of deciding whether a k-hypergraph $(G, \mathcal{H})$ admitting G_L as its automorphism group and defined via a set of orbit representatives $\{1_G, h_{2,i}, h_{3,i}, \ldots, h_{k,i}\}$, $1 \leq i \leq s$, is a k-HRR

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- All one needs to do is to determine the identity fixing automorphisms of $(V, \mathcal{H}')$
- Both V and $\mathcal{H}'$ are considerably smaller than G and $\mathcal{H}$

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- The resulting bipartite vertex-hyperedge incidence graph has $7 + 6 = 13$ vertices, and it is easy to calculate that the stabilizer of the vertex 0 is trivial
- **Thus, a graph on 13 vertices provides the proof for finite cyclic groups of arbitrarily large orders**
- The speed-up is particularly useful when considering small numbers of orbits; which seems to be the case, based on computational results of Mihálová

Thank you