From Equitable Partitions to Full Spectra: A Framework for Symmetric Graphs

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Outline

- Part 1: Spectra of symmetric graphs
- Part 2: Graph with the same spectral radius

Both exploit equitable partitions.

Motivation

Why do we study equitable partitions?

- Spectral graph theory analyzes graph properties (e.g., connectivity, bipartiteness etc.) via eigenvalues and eigenvectors of matrices assigned to a graph (such as adjacency or Laplacian).
- These matrices are real symmetric, hence diagonalizable, but direct diagonalization is computationally expensive for large graphs.
- Equitable partitions allow us to compute *some* eigenvalues exactly without full diagonalization, by reducing the graph to a smaller quotient matrix.
- For highly symmetric graphs, combining equitable partitions with group actions and representations of groups can yield the exact full spectrum efficiently.

A possible shortcut: For symmetric graphs, combine equitable partitions with group actions/representations to obtain the full spectrum.

Equitable Partitions

- Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix, that is given as a block matrix according to a partition $\Pi = (X_1, \ldots, X_m)$ of $\{1, \ldots, n\}$, where the sets X_i, $i = \overline{1, m}$ are nonempty, disjoint and $X_1 \cup \ldots \cup X_m = \{1, \ldots, n\}$ with blocks $A_{ij} \in \mathbb{R}^{|X_i|\times|X_j|}$,

$$A = \left(\begin{array}{c|c|c} A_{11} & \cdots & A_{1m} \\ \hline \vdots & & \vdots \\ \hline A_{m1} & \cdots & A_{mm} \end{array}\right).$$

- $A(\Pi) \in \mathbb{R}^{m\times m}$ denotes the *quotient* matrix and its entries are the average row sums of the blocks A_{ij} of A.
- We say that the partition Π induces an *equitable* partition of A if in every block of A the sum of elements in each row (column) is constant. In case Π is equitable partition $A(\Pi)$ is also called a *divisor* matrix.

Benefits and Drawbacks of Equitable Partitions

Benefits: $A(\Pi)$ is usually of significantly smaller size than A.

Theorem

i) If Π is an equitable partition, then the characteristic polynomial of the quotient matrix $A(\Pi)$ divides that of the adjacency matrix A.

ii) The largest eigenvalue and all *main* eigenvalues of A are also eigenvalues of $A(\Pi)$. (An eigenvalue λ is *main* if its eigenspace is not orthogonal to the all-ones vector $\mathbf{j}$.)

Any eigenvector of $A(\Pi)$ can be lifted up to an eigenvector of A.

Drawback: $A(\Pi)$ may be non-symmetric even if A is symmetric.
Normalize with

$$P = \operatorname{diag}\left(\frac{1}{\sqrt{|X_1|}}, \ldots, \frac{1}{\sqrt{|X_m|}}\right),$$

so that $P^{-1}A(\Pi)P$ becomes symmetric.
$A(\Pi)$ gives some, not all eigenvalues.

Symmetric Graphs - Simple Case

If the adjacency matrix of a graph G is of the form

$$A(G) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad A = A^\top,\ B = B^\top,$$

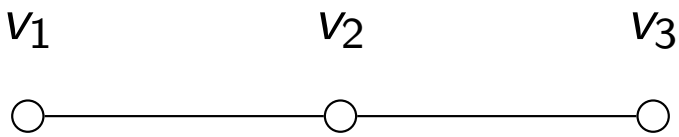
then $\sigma(A(G)) = \sigma(A+B) \cup \sigma(A-B)$.

Example

Let $G = P_2$, then $A(G) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A - B = (-1), A + B = (1)$.

It will also work for any path P_{2k}, $k > 1$.

Symmetric Graph with a Line of Symmetry



Example

Let $G = P_3$, $V(G) = \{v_1, v_2, v_3\}$. Then $U = \{v_1, v_3\}$, $V = \{v_2\}$ is an equitable partition with a divisor matrix

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

and eigenvalues $\{-\sqrt{2}, \sqrt{2}\}$. The remaining eigenvalue is 0, with an eigenvector $(1, 0, -1)^\top$.

What is the Composition Principle?

Heilbronner's composition principle

Heilbronner's composition principle is a **graphical method** for factorizing the characteristic polynomial of symmetric graphs (especially trees) by exploiting their symmetries.

How does it work?

- Break down a large symmetric graph into smaller **fragments**, using a symmetry.
- The eigenvalues of fragments combine to give the full spectrum.
- Uses equitable partitions although Heilbronner did not use this name.

It dates back to 1954. The concept was first introduced by Swiss chemist Edgar Heilbronner, but it was largely ignored by mathematicians. It was later rediscovered in 1974 by McClelland, who provided the first mathematical framework for the method. The concept was replicated, explored, extended, and refined, primarily in problems in chemistry.

The Two Types of Foldings

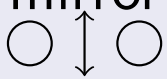
Heilbronner identified two fundamental operations for exploiting symmetry:

Positive Folding

$\oplus$ Symmetric combination

Symmetric partners combine with $+1$ coefficient

mirror



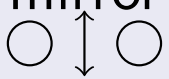
Preserves original eigenvalues

Negative Folding

$\ominus$ Anti-symmetric combination

Symmetric partners combine with -1 coefficient

mirror



Produces another matrix with graph's eigenvalues.

$$Spec(G) = Spec(G^+) \cup Spec(G^-)$$

A Simple Example: Symmetric Tree

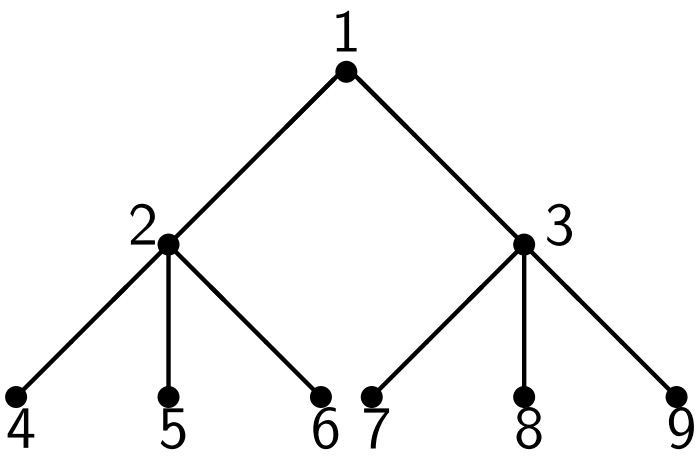


Fig. 1

Canonical equitable partition:

$$V_1 = \{1\} \quad \text{(root)}$$
$$V_2 = \{2, 3\} \quad \text{(level 2)}$$
$$V_3 = \{4, 5, 6, 7, 8, 9\} \quad \text{(level 3)}$$

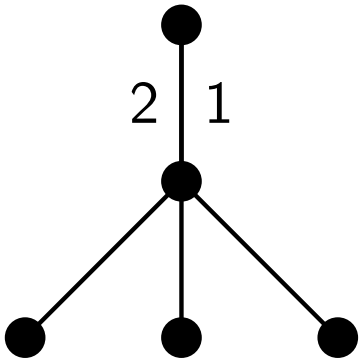
Divisor matrix:

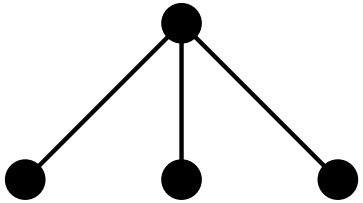
$$D_\Pi = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

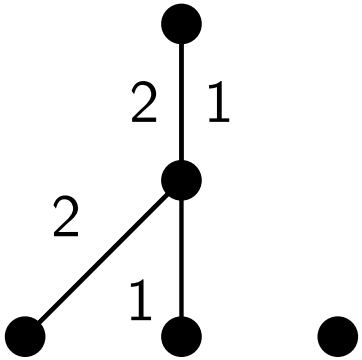
$$\phi(x) = x^5(x - \sqrt{5})(x + \sqrt{5})(x - \sqrt{3})(x + \sqrt{3})$$

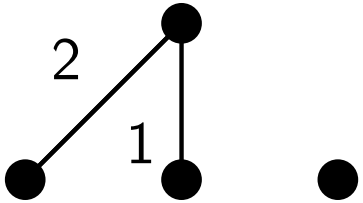
$$\phi_{D_\Pi} = x(x - \sqrt{5})(x + \sqrt{5})$$

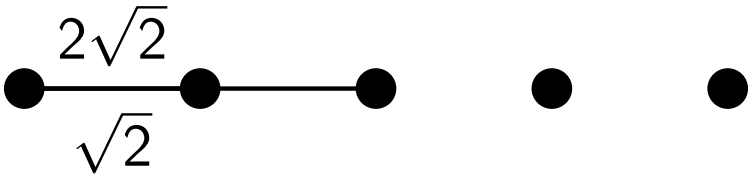
Through Foldings

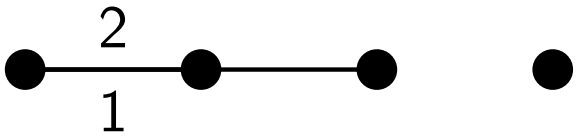












$\pm\sqrt{5},\ 0^3$

$\pm\sqrt{3},\ 0^2$

General Strategy for Type 1 Trees

For a rooted symmetric tree with branching parameters $r_1, r_2, \ldots, r_m$:

- **Identify symmetry levels** vertices at same depth with identical degrees form equitable cells.
- **Construct divisor matrices** D_{Π_ℓ} for each level ℓ:

$$D_{\Pi_\ell} = \begin{pmatrix} 0 & r_\ell & & & \\ 1 & 0 & r_{\ell+1} & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & r_m \\ & & & 1 & 0 \end{pmatrix}$$

- **The characteristic polynomial factorizes**:

$$\phi(x) = \prod_{\ell=1}^{m} \phi_{\Pi_\ell}(x)^{m_\ell}$$

where $m_\ell = (r_{\ell+1} - 1) r_{\ell+2} \cdots r_m$ (with $r_{m+1} = 1$)

Type 2 Trees: Joining Two Symmetric Trees by an Edge

When you join two symmetric trees by an edge, spectrum splits into:

- Spectrum of the branch with an added positive loop.
- Spectrum of the branch with an added negative loop.

Positive Folding

$$D^{+}_{\Pi_m} = \begin{pmatrix} 1 & r_1 & 0 & \cdots \\ 1 & 0 & r_2 & \\ 0 & 1 & 0 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

Negative Folding

$$D^{-}_{\Pi_m} = \begin{pmatrix} -1 & r_1 & 0 & \cdots \\ 1 & 0 & r_2 & \\ 0 & 1 & 0 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

Factorization for Type 2

$$\phi(x) = \phi^{+}_{\Pi_m}(x) \cdot \phi^{-}_{\Pi_m}(x) = \prod_{\ell=1}^{m-1} \phi_{\Pi_\ell}(x)^{2(r_{\ell+1}-1)r_{\ell+2}\cdots r_m}$$

Example: The (p, q)-Double Sun

Consider two stars (p-sun and q-sun) connected by a bridge with distinct number of lengths for the sunrays of each branch, say m and n.

The characteristic polynomial factorizes as:

$$\phi_{\Pi}(x) \cdot (U_m(x/2))^{p-1} \cdot (U_n(x/2))^{q-1}$$

where U_n are Chebyshev polynomials of the second kind.

- $\phi_{\Pi}(x)$ comes from the positive folding along the bridge.
- The Chebyshev factors come from the paths (sunrays) in negative foldings.

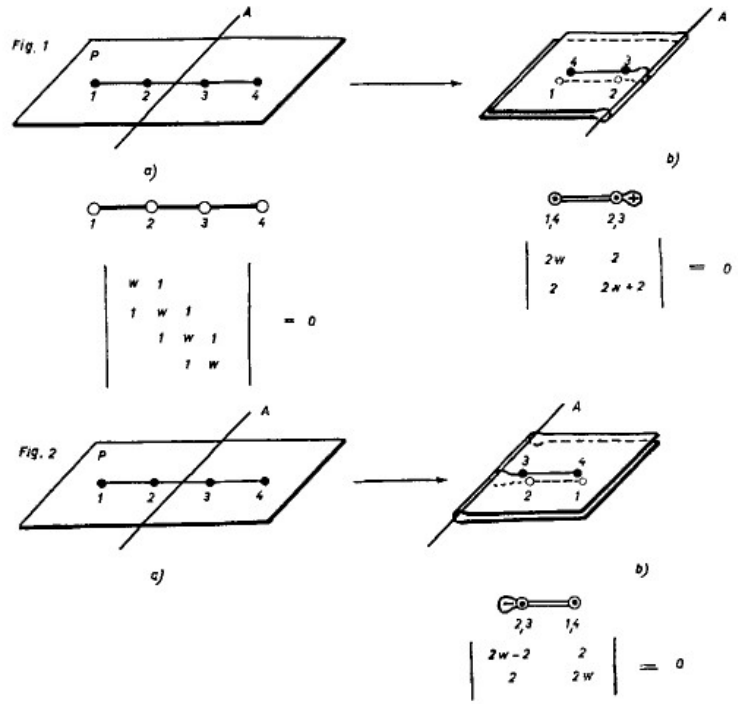
Applications

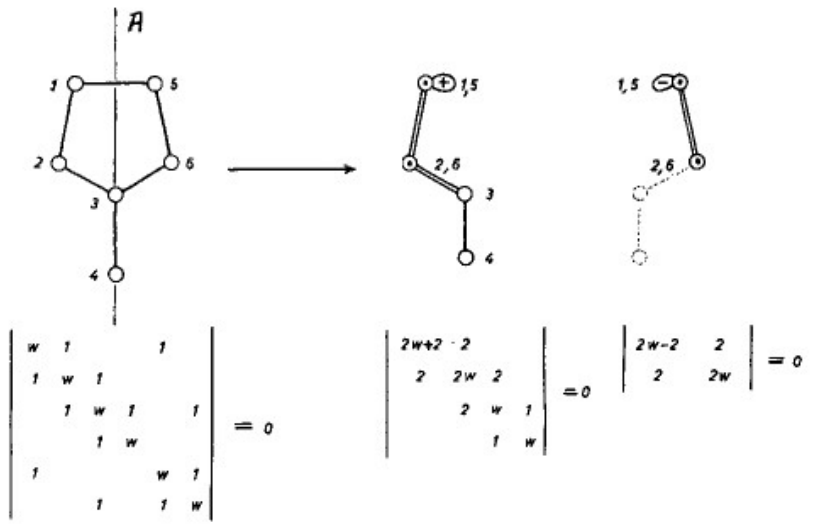
Advanteges of the Composition Principle

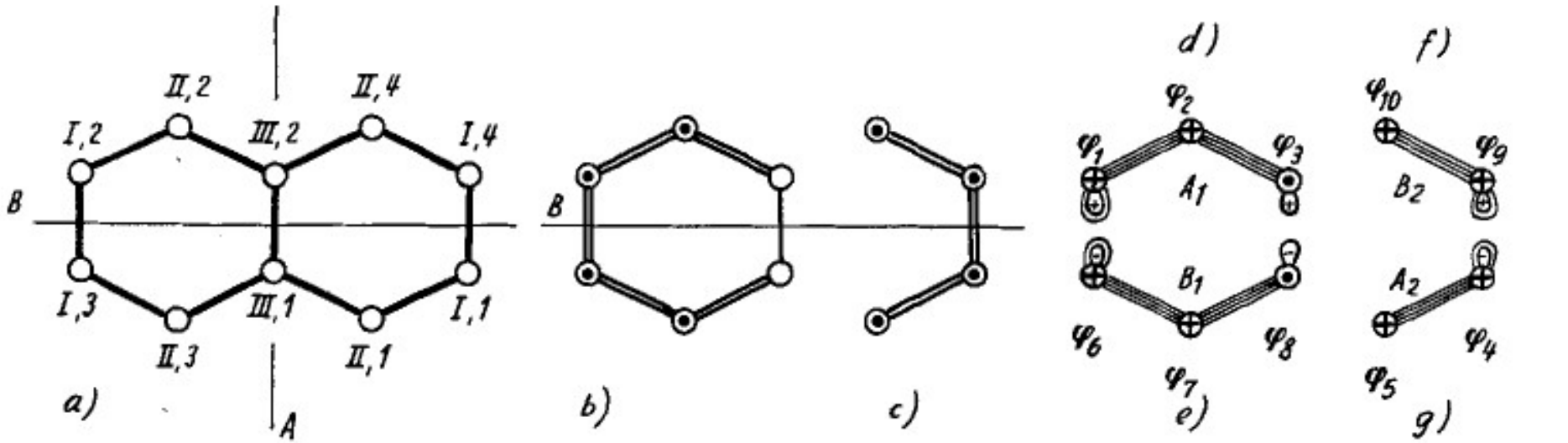
- **Compute spectra of large symmetric trees** using only small divisor matrices.
- **Understand eigenvalue multiplicities** from combinatorial factors.
- **Factorize characteristic polynomials** without solving large determinants.
- **Chemical applications** molecular orbital theory for hydrocarbons.

- **Identify symmetries** group symmetric vertices into equitable cells.
- **Construct matrices** positive and negative foldings.
- **Factorize** - characteristic polynomial $=$ product of polynomials with appropriate multiplicities.
- **Read off eigenvalues** - from smaller, simpler matrices.

From Heilbronner's Paper 1954







Theoretical Background

Main idea

If a graph G has symmetry group Γ, we can fold it into a smaller graph (or matrix) while preserving spectral information. Consider also other types of symmetries not only reflections.

Goal: Compute eigenvalues of A (full spectrum) from smaller folded matrices Q_ρ.
Tool: Group actions, representation theory of groups.

Aut G acting on the vertex set

Let $\Gamma = Aut\ G$. We consider:
Action: $\Gamma \times V(G) \to V(G)$, $g \cdot v = g(v)$ satisfying:

$$e \cdot v = v, \quad (gh) \cdot v = g \cdot (h \cdot v)$$

Adjacency is preserved:

$$(g \cdot u, g \cdot v) \in E(G) \text{ iff } (u, v) \in E(G)$$

Orbit:

$$O_v = \{g \cdot v : g \in \Gamma\}$$

Stabilizer, elements fixing v:

$$\Sigma_v = \{g \in \Gamma : g \cdot v = v\}$$

Orbit-Stabilizer Theorem: $|\Gamma| = |O_v||\Sigma_v|$
Fixed points of g: Vertices not moved under the groups action.

$$Fix(g) = \{v : g \cdot v = v\}$$

$$\chi_{\text{perm}}(g) = |Fix(g)|$$

Consequence

Orbits partition $V(G)$: $V(G) = O_1 \sqcup O_2 \sqcup \cdots \sqcup O_k$ is always inducing an equitable partition.

Example

- $G = P_3$
- $V(G) = \{v_1, v_2, v_3\}$, $\Gamma = Aut\, G = \mathbb{Z}_2 = \{e, \tau\}$
- Orbits: $O_1 = \{v_2\}$, $O_2 = \{v_1, v_3\}$
- Stabilizers: $\Sigma_{v_1} = \{e\}$, $\Sigma_{v_2} = \Gamma$
- Fixed points: $Fix(e) = \{v_1, v_2, v_3\}$, $Fix(\tau) = \{v_2\}$

What Group Actions Alone Give?

We get quotient graph (one folding):

- Vertices are orbits.
- Edges are connections between orbits. They can be weighted.

So, we need something else ...

Representations - Basics

- A homomorphism $\rho : \Gamma \to GL(d, \mathbb{C})$ assigns a matrix to each group element.
- An irreducible representation is a homomorphism $\rho : \Gamma \to GL(d, \mathbb{C})$ with no nontrivial invariant subspaces.
- Finite group Γ has finitely many irreducible representations over $\mathbb{C}$.
- $\sum_\rho (\dim \rho)^2 = |\Gamma|$.
- Characters $\chi(g) = \mathrm{tr}(\rho(g))$ determine ρ uniquely. Two representations are equivalent iff they have the same character.

Example

$\Gamma = \mathbb{Z}_3 = \langle r \rangle, w = e^{2\pi i/3}$

$$\rho_0(r) = 1, \quad \rho_1(r) = w, \quad \rho_2(r) = w^2.$$

Permutation Representation

The permutation representation $\pi : \Gamma \to GL(n, \mathbb{C})$ maps g to its permutation matrix $P(g)$:

$$(P(g))_{ij} = \begin{cases} 1 & \text{if } g \cdot v_i = v_j \\ 0 & \text{otherwise} \end{cases}$$

Properties:

- $\pi(e) = I_n$
- $\pi(gh) = \pi(g)\pi(h)$
- $\pi(g^{-1}) = \pi(g)^\top$
- Permutation matrices $P(g)$ satisfy $P(g)A = AP(g)$

Character:

$$\chi_{\text{perm}}(g) = \text{tr}(\pi(g)) = \#\{v \in V : g \cdot v = v\}$$

Decomposition into Isotypic Components

Theorem

The permutation representation decomposes into $\bigoplus_{\rho \in \hat{\Gamma}} m_\rho \cdot \rho$ *where* m_ρ *is the multiplicity of irrep* ρ*.*

Multiplicity formula

$$m_\rho = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{\text{perm}}(g)\, \overline{\chi_\rho(g)}$$

Each isotypic component V_ρ has dimension $m_\rho \cdot \dim \rho$ and is invariant under A, $AV_\rho \subset V_\rho$. In symmetry adapted basis A has block diagonal form

$$U^{-1}AU = \begin{pmatrix} \text{block for } \rho_0 & & \\ & \text{block for } \rho_1 & \\ & & \ddots \end{pmatrix}$$

Folding means constructing each block without building the full basis.

Folding: The Quotient Method

Given orbits $O_1, \ldots, O_k$ and representatives $r_p \in O_p$, for irrep ρ of dimension d, the folded matrix Q_ρ is a $k \times k$ block matrix where each block is $d \times d$:

$$(Q_\rho)_{pq} = \frac{1}{|\Sigma_{r_p}|} \sum_{g \in \Gamma} \rho(g) E_{pq}(g),$$

where

$$E_{pq} = \begin{cases} 1 & (r_p, g(r_q)) \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

For 1-dimensional irreps, each $\rho(g)$ is a scalar.
Regular action: $|O_{r_p}| = |\Gamma|$ for all r_p.

Example: P_3 with $\mathbb{Z}_2$

Vertices: v_1, v_2, v_3. Reflection τ swaps $v_1 \leftrightarrow v_3$, fixes v_2.
Irreps of $\mathbb{Z}_2$: $\rho_0(\tau) = 1,\ \rho_1(\tau) = -1$
Permutation character: $\chi_{\text{perm}}(e) = 3,\ \chi_{\text{perm}}(\tau) = 1$
Multiplicities:

$$m_{\rho_0} = \frac{1}{2}(3 \cdot 1 + 1 \cdot 1) = 2, \quad m_{\rho_1} = \frac{1}{2}(3 \cdot 1 + 1 \cdot (-1)) = 1$$

Thus π decomposes as $2\rho_0 \oplus 1\rho_1$.
For ρ_0 (trivial, $\rho_0(g) = 1$):

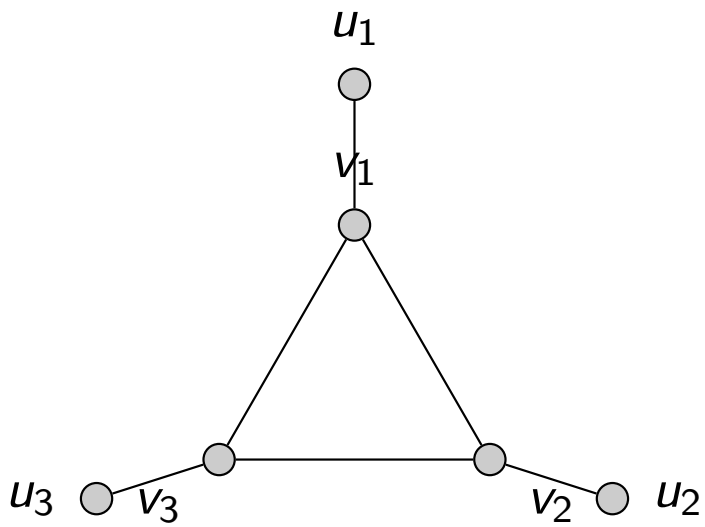
$$Q_{\rho_0} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues: $\pm\sqrt{2}$
For ρ_1 (sign, $\rho_1(\tau) = -1$):

$$Q_{\rho_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Projection gives one zero eigenvalue.
Full spectrum: $\{\sqrt{2}, -\sqrt{2}, 0\}$

Example: Sun Graph S_3 (Triangle + Pendants)



Vertices: v_1, v_2, v_3 (triangle) + u_1, u_2, u_3 (pendants attached to v_i)
$\Gamma = \mathbb{Z}_3$ rotates both orbits regularly: $O_1 = \{v_1, v_2, v_3\}$, $O_2 = \{u_1, u_2, u_3\}$, $|O_{1/2}| = 3 = |\Gamma|$

Folded matrices ($k = 2$):

$$Q_{\rho_0} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_{\rho_1} = Q_{\rho_2} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues: - $\rho_0 : 1 \pm \sqrt{2}$ (multiplicity 1 each) - $\rho_1, \rho_2 : \frac{-1 \pm \sqrt{5}}{2}$ (each appears twice). Spectrum: $\{1 + \sqrt{2},\ 1 - \sqrt{2},\ \frac{-1+\sqrt{5}}{2}^{(2)},\ \frac{-1-\sqrt{5}}{2}^{(2)}\}$

How to Get Multiplicities

Regular action case (all orbits size $|\Gamma|$)

$$\mathrm{mult}_A(\lambda) = \dim \rho \cdot \mathrm{mult}_{Q_\rho}(\lambda)$$

Each eigenvalue of Q_ρ appears with multiplicity equal to $\dim \rho$.

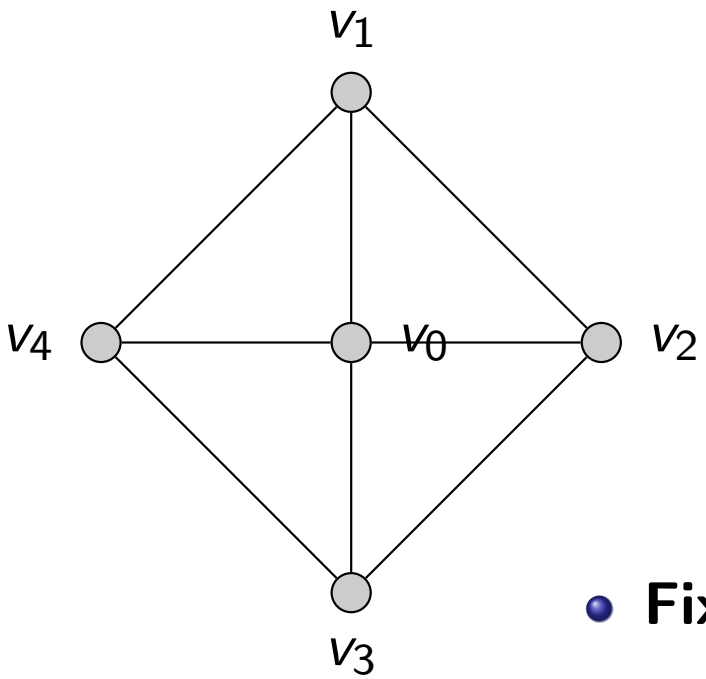
General case

- Compute m_ρ from character formula.
- ρ-isotypic component V_ρ has dimension $m_\rho \cdot \dim \rho$.
- Project canonical vectors onto V_ρ using:

$$P_\rho = \frac{\dim \rho}{|\Gamma|} \sum_{g \in \Gamma} \overline{\chi_\rho(g)}\, \pi(g)$$

- Those give eigenvectors.

Wheel Graph W_4



- $Aut\ G = \mathbb{D}_4$
- $O_1 = \{v_0\}; O_2 = \{v_1, v_2, v_3, v_4\}$
- Irreducible representations: $\{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\}$
- Character table for $\mathbb{D}_4$

Class	$\{e\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
Size	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

- **Fixed vertices:**
  - $\chi_{\text{perm}}(e) = 5$
  - $\chi_{\text{perm}}(r^2) = 1$
  - $\chi_{\text{perm}}(r) = \chi_{\text{perm}}(r^3) = 1$
  - $\chi_{\text{perm}}(s) = \chi_{\text{perm}}(sr^2) = 3$
  - $\chi_{\text{perm}}(sr) = \chi_{\text{perm}}(sr^3) = 1$

- **Multiplicities of irreps:**

$$m_{\rho_0} = 2,\ m_{\rho_1} = 0,\ m_{\rho_2} = 1,\ m_{\rho_3} = 0,\ m_{\rho_4} = 1$$

- **Folded matrices:**

$$Q_{\rho_0} = \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix},\ Q_{\rho_1} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix},\ Q_{\rho_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- **Projections to get correct multiplicities:**

$$P_{\rho_2}(e_{v_0}) = 0, P_{\rho_2}(e_{v_i}) = \frac{1}{4}(e_{v_1} + e_{v_3} - e_{v_2} - e_{v_4})$$

For $v = e_{v_1} + e_{v_3} - e_{v_2} - e_{v_4}, \quad Av = -2v.$

Similarly,

- $P_{\rho_4}(e_{v_0}) = 0, P_{\rho_4}(e_{v_1}) = \frac{1}{2}(e_{v_1} - e_{v_3}),\ P_{\rho_4}(e_{v_2}) = \frac{1}{2}(e_{v_2} - e_{v_4}), P_{\rho_4}(e_{v_3}) = -\frac{1}{2}(e_{v_1} - e_{v_3}), P_{\rho_4}(e_{v_3}) = -\frac{1}{2}(e_{v_2} - e_{v_4})$
- $e_{v_1} - e_{v_3}, e_{v_2} - e_{v_4}$ are eigenvectors of A and corresponding eigenvalue is 0.

Complete Graph K_n with S_n

One orbit (transitive). Permutation representation decomposes as:

$$\rho_0 \oplus \rho_{\text{std}}$$

where ρ_0 is trivial (dim 1), ρ_{std} is standard (dim $n-1$).
Folded matrices:

$$Q_{\rho_0} = (n-1), \qquad Q_{\rho_{\text{std}}} = (-1) \cdot I_{n-1}$$

Spectrum:

$$\sigma(K_n) = \{n-1\} \cup \{-1\}^{(n-1)}$$

Reduced $n \times n$ problem to two scalars.

Folding Algorithm

- Find orbits $O_1, \ldots, O_k$ of Γ on V.
- Compute permutation character $\chi_{\text{perm}}(g)$ for all $g \in \Gamma$.
- For each irrep ρ of Γ:
  - Compute multiplicity m_ρ (character formula).
  - If $m_\rho = 0$, skip.
  - Construct folded matrix Q_ρ (sum over group with $\rho(g)$).
  - Diagonalize Q_ρ or find its eigenvalues.
  - Determine multiplicities (regular case: $\dim \rho \times$ multiplicity in Q_ρ).
- Combine eigenvalues from all ρ with multiplicities.

When Is Folding Useful?

Good cases:

- Nontrivial automorphism group;
- Regular actions;
- Vertex-transitive graphs;
- Cayley graphs;
- Distance-regular graphs.

Not so good:

- Asymmetric graphs (Γ trivial)
- Many irreps.

Too Large Groups

In case of symmetric trees automorphism group can be too large. At each level there are many permutations that preserve adjacency. In this case if is better to apply folding method to a suitable subgroups. Heilbronner's composition principle relies on $\mathbb{Z}_2$.

Another Application of Equitable Partitions - Constructing Infinite Families of Graphs with the Same Spectral Radius

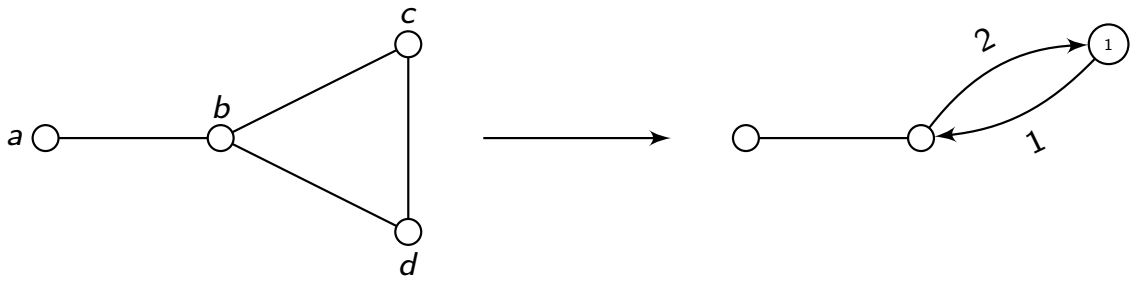


Figure: The paw graph and a divisor graph.

For the *paw graph*, we may consider the equitable partition $\Pi : U \sqcup V$, with

$$U = \{a\}, \quad V = \{b\}, \quad W = \{c, d\}.$$

The divisor matrix is

$$A_\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Motivation comes from

- A. Seeger, D. Sossa, Infinite families of connected graphs with equal spectral radius, Australas. J. Combin. 87 (2023) 258–276.

Same Spectral Radius

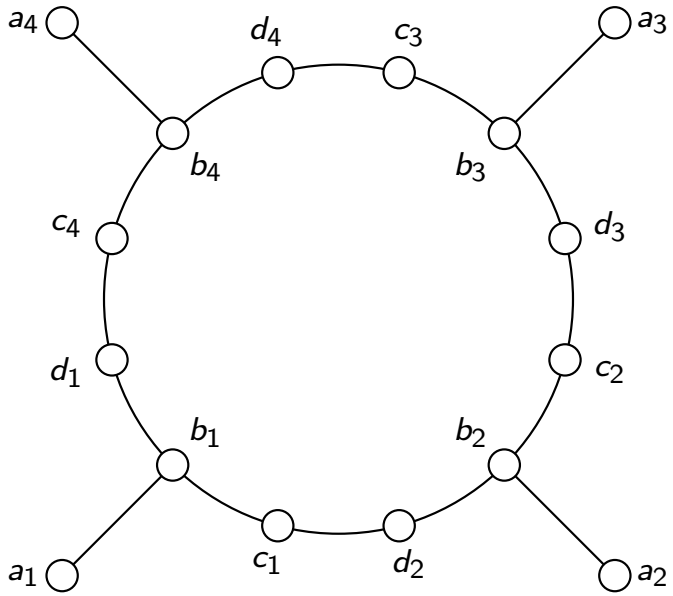


Figure: The graph U_4

$$U = \{a_i : 1 \leq i \leq 4\},\ V = \{b_i : 1 \leq i \leq 4\},\ W = \{c_i, d_i : 1 \leq i \leq 4\}.$$

$$A_\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Algorithm

Step 1. Start with a graph G_1, with at least one cycle, whose vertices are partitioned as follows.

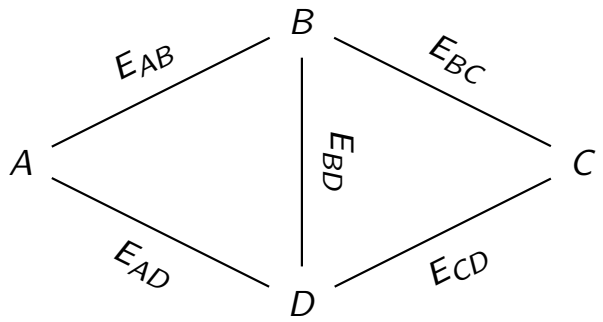


Figure: A partitioned graph G_1.

E_{AB} represents the sets of all edges connecting the vertices of A with the vertices of B. These vertices are called *leading vertices* of the corresponding partition.

Step 2. From the graph G_1, create the graph $\tilde{G}_1$, by adding another copy of vertices in C that are connected by E_{CD} to all vertices in D. We assume that the existing set of vertices C remains connected to the vertices in B by E_{BC}.

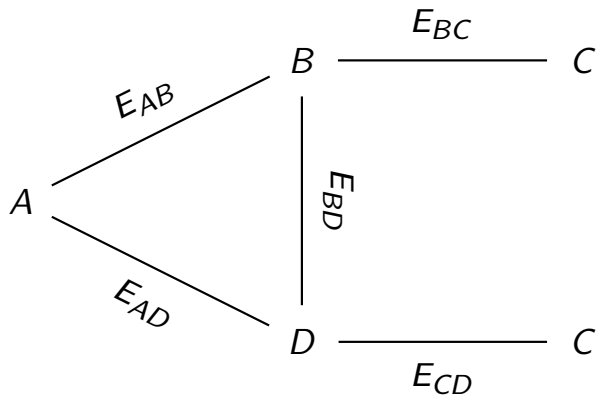


Figure: The partitioned graph $\tilde{G}_1$.

Step 3. Create a number of copies of the graph $\tilde{G}_1$.

Step 4. To the upper set of vertices C of the partition of $\tilde{G}_1$, connect a copy of $\tilde{G}_1$ by joining the leading vertices of that set C to the leading vertices of D determined by the edges defined by E_{CD}. Proceed in a similar way with the remaining copies. If the number of copies is p, the resulting graph is denoted by G_p.

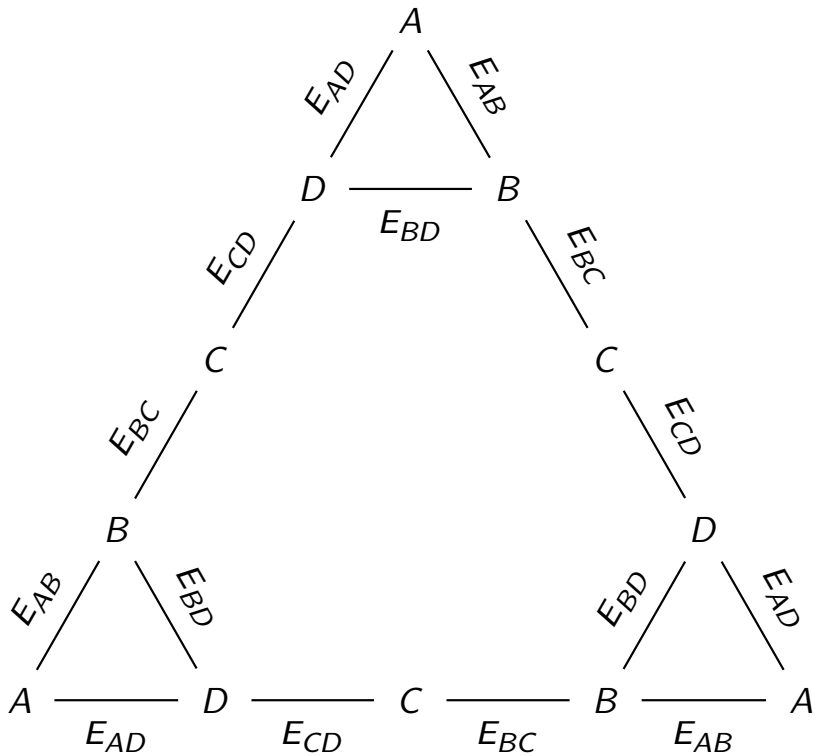


Figure: The graph G_3.

Main Theorem

Theorem

The graphs G_p, for $p \in \{1, 2, \ldots\}$, share the same spectral radius.

Example

We begin with the paw graph. In Step 2, we obtain the graph $\tilde{U}_1$.

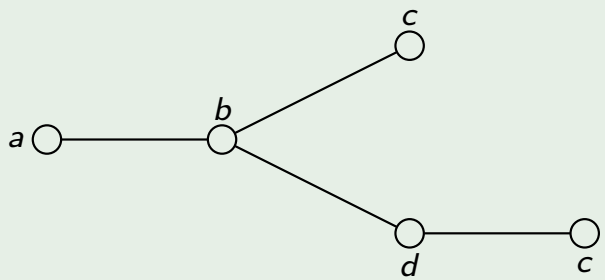


Figure: The graph $\tilde{U}_1$.

In Step 3, we create four copies of $\tilde{U}_1$. Finally, we connect the copies according to pattern given in U_1. The resulting graph U_4 shares the spectral radius with U_1.

More Elaborated Examples

Example

We consider a more elaborated graph.

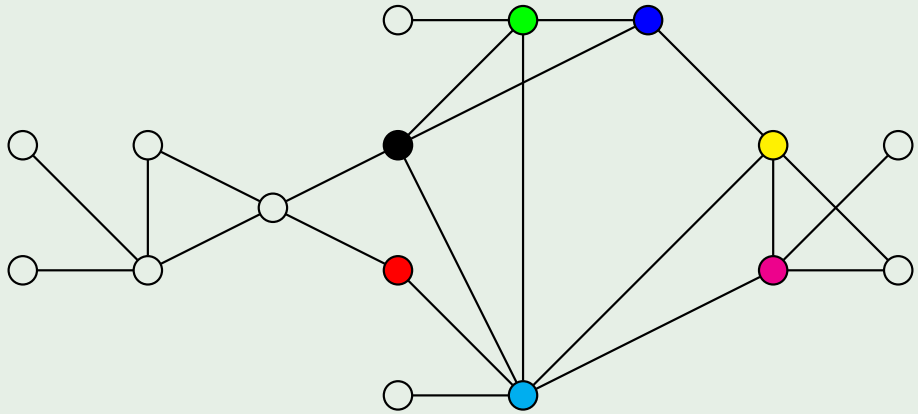


Figure: The graph H with coloured leading vertices.

In this case the partition A, B, C, D is constituted by the set of vertices whose leaders are, respectively, black and red, green and blue, yellow and magenta, and cyan.

Example

If we apply the previous algorithm once, we obtain the graph H_2.

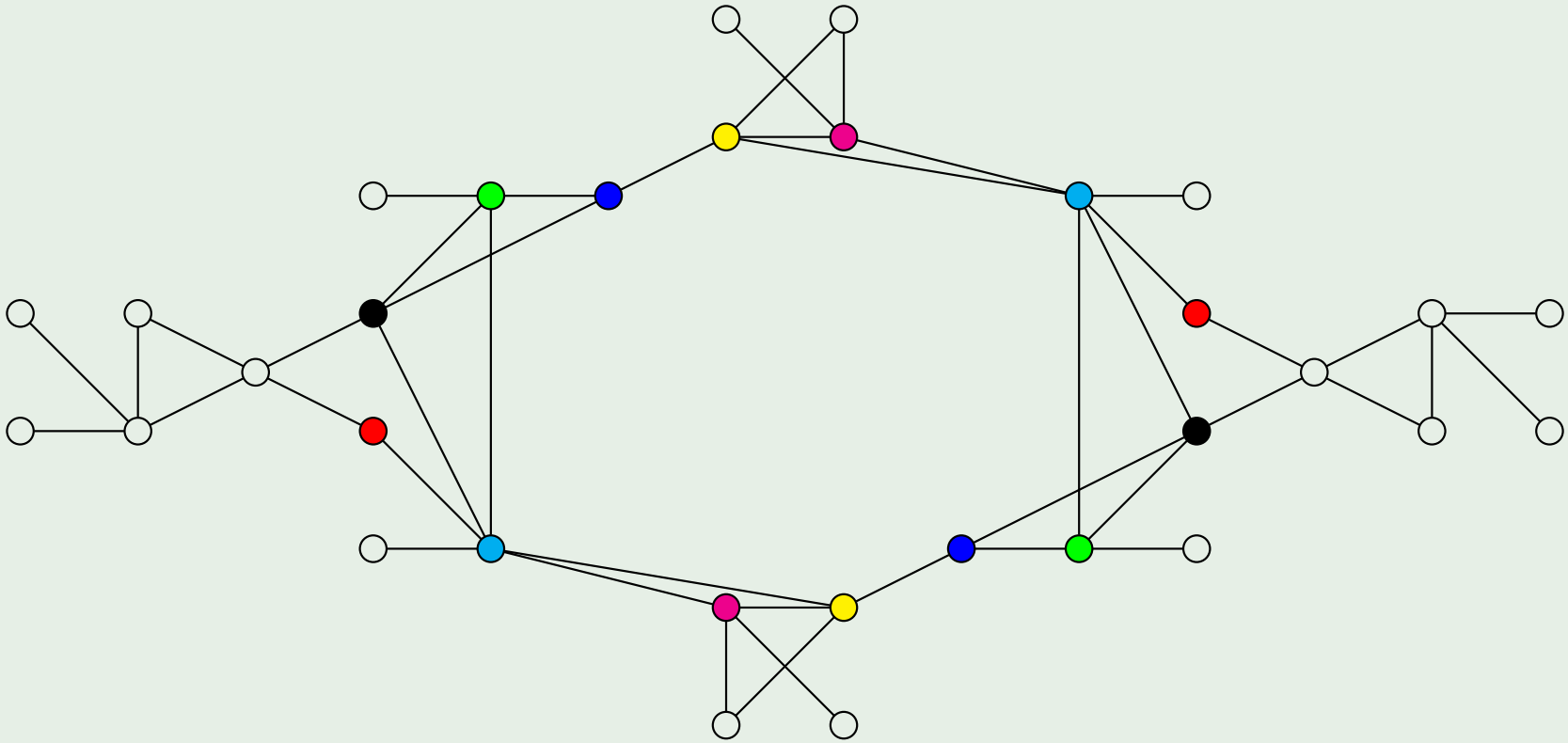


Figure: Graph resulting from one iteration of the algorithm over H.

Both graphs have index 3.58102.

Non Isomorphic Families of Graphs with the Same Spectral Radius

An interesting aspect of the presented method is the fact that one can construct different families of connected graphs depending on the initial partition A, B, C, D.

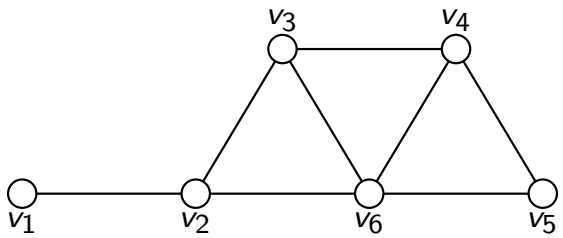


Figure: A graph of order 6.

We consider the vertex-partition

$$A = \{v_1, v_2\}$$
$$B = \{v_3, v_4\}$$
$$C = \{v_5\}$$
$$D = \{v_6\}.$$

First Iteration

One iteration of our construction leads to the graph:

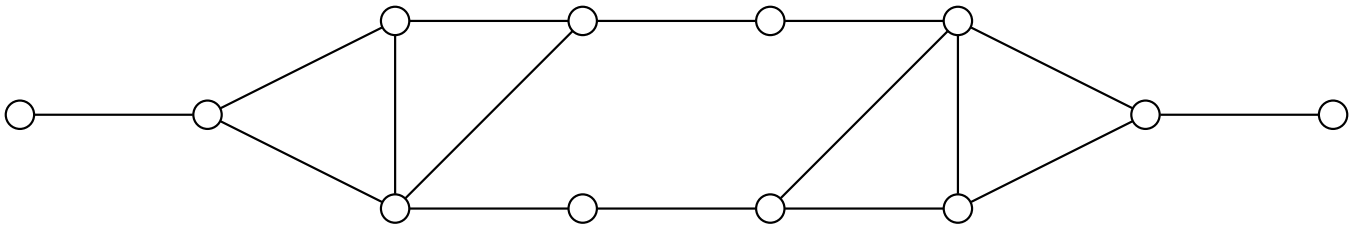


Figure: A possible iteration.

Second Iteration

However, if we consider the partition

$$A = \{v_1, v_2\}$$
$$B = \{v_3\}$$
$$C = \{v_4, v_5\}$$
$$D = \{v_6\}.$$

then we obtain the following graph.

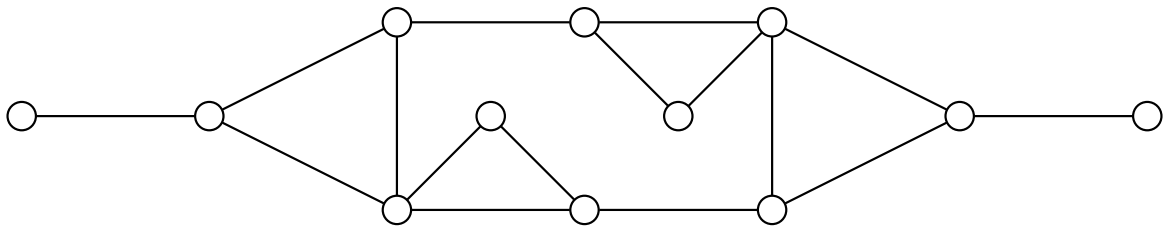


Figure: Another possible iteration.

These graphs share the spectral radius 2.98094 with the initial graph. Clearly, they are not isomorphic.

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Concluding Remark

Equitable partitions - small matrices, big insights.

Thank you!

Hvala!