Metric Dimension of Some Structured Graph Families: Exact Values and Explicit Bases

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Outline

Metric Representation (Slater 1975; Harary and Melter 1976)

Let G be a connected graph, and let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices in G. The **metric representation** of a vertex $v \in V(G)$ with respect to W is the k-vector:

$$r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

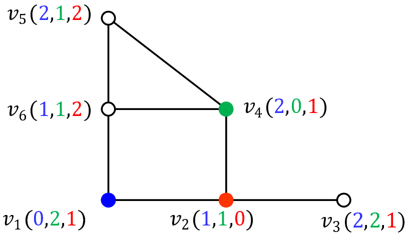
where $d(x, y)$ denotes the graph distance between vertices x and y.

Terminology

Metric Dimension (Slater 1975; Harary and Melter 1976)

- A set W is called a **resolving set** for G if all vertices of G have distinct metric representations with respect to W.
- Members of a resolving set are referred to as **landmarks**.
- A resolving set of minimum cardinality is called a **basis** for G.
- The number of vertices in a basis is the **metric dimension** of G, denoted $\dim(G)$.
- If $\dim(G) = k$, then G is called a k-**dimensional graph**.

Example: Warm-Up



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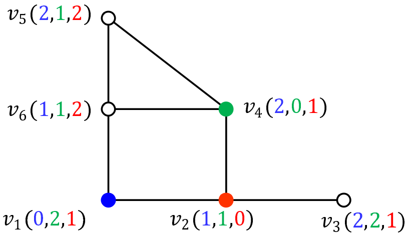
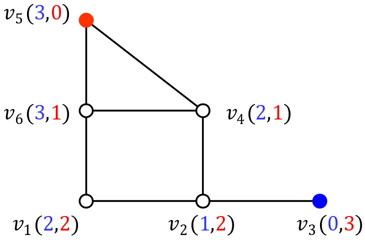
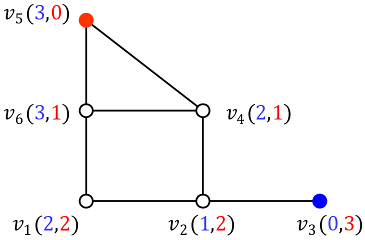


$$\dim(G) \leq 3$$

Example: Warm-Up

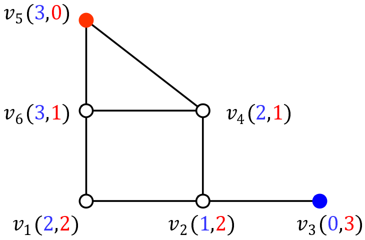


Example: Warm-Up



$$\dim(G) \leq 2$$

Example: Warm-Up



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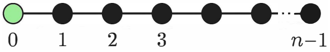

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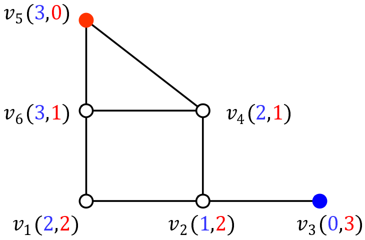
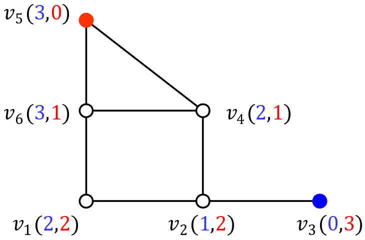


$$\dim(G) \leq 2$$

Theorem (Chartrand et al. 2000)

Let G be a connected graph. Then $\dim(G) = 1$ iff G is a path.

Example: Warm-Up



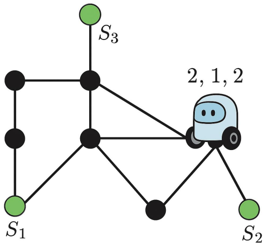
$$\dim(G) \leq 2$$
$$\dim(G) = 2$$

Theorem (Chartrand et al. 2000)

Let G be a connected graph. Then $\dim(G) = 1$ iff G is a path.

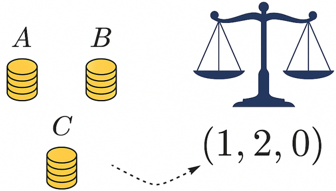
Application: Robot Navigation (Khuller and Raghavachari 1996)

- A moving point (e.g., a robot) in a graph can determine its location by measuring distances to a set of fixed sonar stations.
- The goal is to find a minimum set of labeled vertices such that the robot can uniquely identify its current position.
- Such a set of vertices forms a resolving set.



Application: Coin Weighing (Sebo & Tannier 2004)

- We have n coins, each either **light** or **heavy**.
- In a single weighing, we select a subset of coins and the scale tells us **how many are light**.
- Goal: Determine whether each coin is light or heavy using the **fewest possible weighings**.
- Static version: all weighing subsets are fixed in advance.



Sketch of the idea (Sebo & Tannier 2004)

- Each coin configuration (e.g., LHL) is a vertex in the **hypercube** Q_n.
- Each weighing subset acts like a **landmark** by returning a number: the count of light coins.
- Each configuration yields a vector of responses: analogous to a **metric code**.
- A set of weighings is valid iff it **uniquely identifies all configurations**, i.e., forms a resolving set in Q_n.
- They prove that the **minimum number of weighings** differs from $\dim(Q_n)$ by at most ± 1.

Application: Mastermind (Cáceres et al. 2007)

A two-player game: codemaker and codebreaker.

- Secret code $s \in \{1, 2, \ldots, k\}^n$ is chosen.
- Guesses $t \in \{1, 2, \ldots, k\}^n$ are submitted by the codebreaker.
- Each guess returns $a(s,t) = |\{i : s_i = t_i\}|$: the number of correct symbols in correct positions.

Sketch of the idea (Cáceres et al. 2007)

- All codes form the vertex set of the **Hamming graph** $H_{n,k} = K_k \square K_k \square \cdots \square K_k$.
- The "distance" $d(s,t) = n - a(s,t)$ is the **Hamming distance**.
- A set of queries $\{t_1, \ldots, t_m\}$ distinguishes all codes iff it is a **resolving set** in $H_{n,k}$.
- Therefore, the **minimum number of static queries** equals $\dim(H_{n,k})$.

Application: Network Discovery (Beerliova et al., 2006)

- In many real networks (e.g., communication, sensor, or social networks), the global topology is initially unknown.
- Goal: Discover the full structure of a connected graph $G = (V, E)$ using limited local information.

Local Map Model:

- Each selected node $v \in V$ reveals all **shortest paths** from v to other vertices.
- The union of all such paths (from selected nodes) gives a partial view of the graph.
- The goal is to select a minimum-size subset $S \subseteq V$ such that the union of local maps from S reveals all of G.

Metric Dimension vs. Network Discovery

- In metric dimension, a resolving set uniquely identifies each vertex by its distances to landmarks.
- In network discovery, a landmark set reveals enough edge information via shortest paths to reconstruct G.

- Both problems rely on the structure of **shortest paths**.
- The resolving set from metric dimension can serve as a heuristic for selecting landmarks in network discovery.
- However, network discovery focuses on edge coverage, not vertex distinguishability.

Goal: Find a minimal set $S \subseteq V$ whose local maps reconstruct the entire graph.

Computational Complexity

- Problem: Given a graph G and integer k, is $\dim(G) \leq k$?
- This problem is NP-complete (Khuller et al. 1996).
- Even remains NP-complete for:
  - Planar graphs (even restricted to bounded degree).
  - Bipartite, split, cobipartite, and line graphs of bipartite graphs.
  - Interval and permutation graphs, even when restricted to diameter 2.
- However, it is solvable in polynomial time for trees and outerplanar graphs.
- Efficient algorithms are known for cographs, chain graphs, and cactus block graphs.
- There is also a $\big(2\log n + \Theta(1)\big)$-approximation algorithm for the metric dimension of every n-vertex graph, based on the classical greedy algorithm for the Set Cover problem.

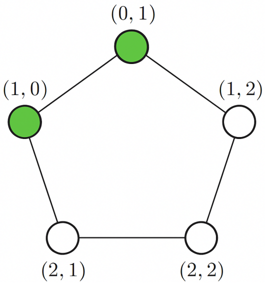
Characterization of Extremal Metric Dimension

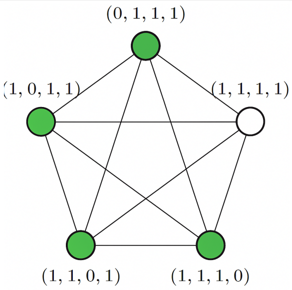
Let G be a connected graph of order $n \geq 1$. Then:

- $\dim(G) = 1$ iff $G = P_n$. Let $\{w\}$ be a basis. For each v, $r(v \mid W) = d(v, w)$, thus there exists u with $d(u, w) = n - 1$, so $G = P_n$.
- $\dim(G) = n - 1$ if and only if $G = K_n$. If $G \neq K_n$, then there exist $u, v, w \in V(G)$ such that $u \sim v$ and $u \nsim w$; hence $V(G) \setminus \{v, w\}$ is a resolving set.
- For $n \geq 4$, $\dim(G) = n - 2$ if and only if:
    - $G = K_{r,s}, \quad r, s > 1$, or
    - $G = K_r \vee \overline{K_s}, \quad r \geq 1, s \geq 2$, or
    - $G = K_r \vee (K_1 \cup K_s), \quad r, s \geq 1$.

Metric Dimension of Some Graphs

- $\dim(K_n) = n - 1$
- $\dim(P_n) = 1$ for $n \geq 2$
- $\dim(C_n) = 2$ for $n \geq 3$
- $\dim(K_{s,t}) = s + t - 2$ if $s, t \geq 2$





Metric Dimension of Hypercubes

Theorem (Erdős and Rényi, 1963)

$$\lim_{n\to\infty} \dim(Q_n) \cdot \frac{\log n}{n} = 2.$$

Theorem (Cáceres et al., 2007)

$$\dim(Q_{n-1}) \leq \dim(Q_n) \leq \dim(Q_{n-1}) + 1$$

n	1	2	3	4	5	6	7	8	9	10	11	15	17
$\dim(Q_n)$	1	2	3	4	4	5	6	6	7	7	≤ 8	≤ 9	≤ 10

Bounds on $\dim(H_{n,k})$

Theorem (Erdős and Rényi, 1963)

$$\dim(H_{n,k}) \geq (2 + o(1)) \frac{n \log k}{\log n}.$$

Theorem (Chvátal, 1983)

For every $\varepsilon > 0$, if n is sufficiently large and $k < n^{1-\varepsilon}$, then

$$\dim(H_{n,k}) \leq (2 + \varepsilon)n \cdot \frac{1 + 2 \log k}{\log n - \log k}.$$

Theorem (Cáceres et al., 2007)

$$\dim(H_{2,k} = K_k \square K_k) = \left\lfloor \frac{2}{3}(2k - 1) \right\rfloor.$$

Why Cartesian products?

- The preceding applications share a common feature: the underlying state space naturally decomposes into several independent coordinates, layers, or choices.
- This structure is captured by the Cartesian product of graphs. In particular,

$$d_{G\Box H}\big((g,h),(g',h')\big) = d_G(g,g') + d_H(h,h').$$

  Thus distances in the product are built from distances in the factors.
- Important examples already appeared in this form:

$$Q_n = \underbrace{K_2\Box\cdots\Box K_2}_{n \text{ times}}, \qquad H_{n,k} = \underbrace{K_k\Box\cdots\Box K_k}_{n \text{ times}}.$$

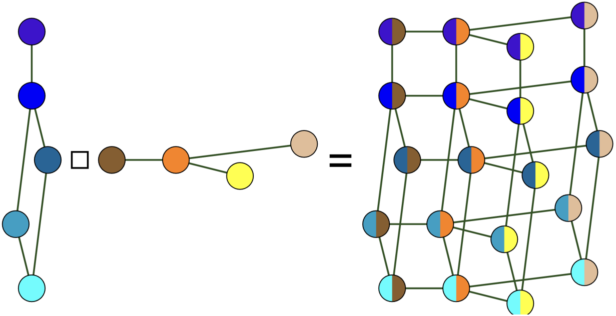
- This leads to a natural structural question: how does resolving information in the factors combine in the product?

Guiding problem

Determine, or sharply estimate, $\dim(G\Box H)$ from the structure of G and H.

Definition

The Cartesian product of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H) := \{(u, v) : u \in V(G), v \in V(H)\}$, where (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.



Bounds on Metric Dimension of Cartesian Products

- (Chartrand et al., 2000): $\dim(G) \leq \dim(G \Box K_2) \leq \dim(G) + 1.$
- (Hernando et al., 2005): For connected graphs G and H,
  $\max\{\dim(G), \dim(H)\} \leq \dim(G \Box H) \leq$
  $\min\{\dim(G) + |V(H)|,\ \dim(H) + |V(G)|\} - 1.$
- Also, for $n \geq 3$:
  $$\dim(G \Box K_n) \leq \dim(G) + n - 2.$$
- Moreover:
  $$\dim(G) \leq \dim(G \Box P_n) \leq \dim(G) + 1$$
  $$\dim(G \Box C_n) \leq \begin{cases} \dim(G) + 1 & \text{if } n \text{ odd,} \\ \dim(G) + 2 & \text{if } n \text{ even.} \end{cases}$$

Exact Values for $\dim(G\square H)$

There are relatively few exact results on $\dim(G\square H)$ due to the complexity of the problem. Known cases include:

- (Cáceres et al., 2007): determined $\dim(G\square H)$ for specific graph pairs $G, H \in \{P_n, C_n, K_n\}$.
- Khuller et al. (1996): $\dim(P_m\square P_n) = 2$ for $m, n \geq 2$.
- Hernando et al. (2005):

$$\dim(C_m\square C_n) = \begin{cases} 3 & \text{if } m \text{ or } n \text{ is odd,} \\ 4 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

- For $m \leq n$:

$$\dim(K_m\square K_n) = \begin{cases} n-1 & \text{if } 2(m-1) < n, \\ \left\lfloor \frac{2(m+n-1)}{3} \right\rfloor & \text{otherwise} \end{cases}$$

- For $n \geq 3$: $\dim(P_m\square K_n) = n-1$.

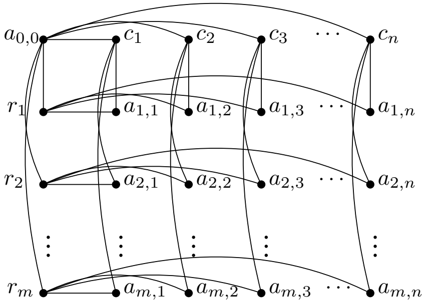
$\dim(K_{1,m}\square K_{1,n}) = ?$

$$\dim(K_{1,n}) = \begin{cases} 1 & \text{if } n \leq 2, \\ n-1 & \text{if } n \geq 3. \end{cases}$$

$\dim(K_{1,m} \square K_{1,n}) = ?$

Let $G := K_{1,m} \square K_{1,n}$, where $m \leq n$. We analyze three cases:

- $m = 1$
- $2 \leq m < \frac{n}{2}$
- $\frac{n}{2} \leq m \leq n$



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Theorem (Davoodi and Jannesari, 2026+)

For $K_{1,1} \square K_{1,n}$:

$$\dim(K_{1,1} \square K_{1,n}) = \begin{cases} 2 & \textit{if } n = 1, \\ n & \textit{if } 2 \leq n \leq 4, \\ n - 1 & \textit{if } n \geq 5. \end{cases}$$

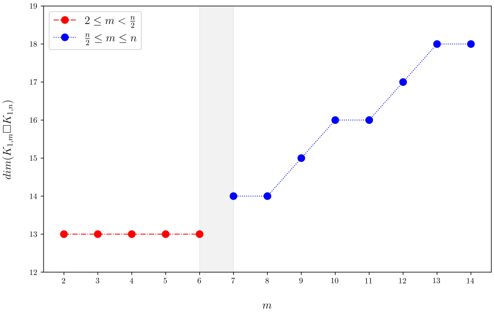
Theorem (Davoodi and Jannesari, 2026+)

If $2 \leq m < \frac{n}{2}$, *then* $\dim(K_{1,m}\Box K_{1,n}) = n - 1$.

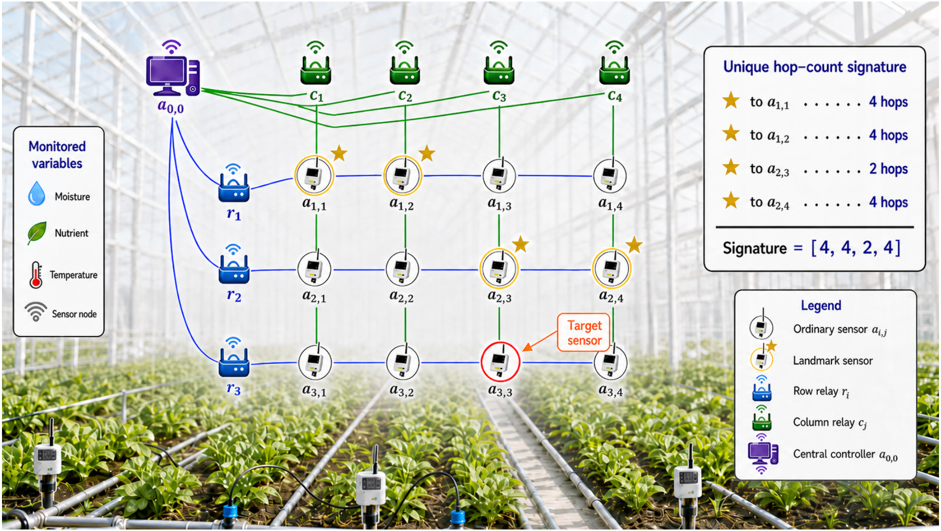
Theorem (Davoodi and Jannesari, 2026+)

If $\frac{n}{2} \leq m \leq n$, *then* $\dim(K_{1,m}\Box K_{1,n}) = n + \lfloor\frac{2m-n}{3}\rfloor$.

Example: $\dim(K_{1,m}\Box K_{1,n})$ as a function of m, for fixed $n = 14$.



Application: precision-agriculture deployment



Precision-agriculture deployment: metric interpretation

Sketch of the idea

- The greenhouse layout is modeled by the hub-and-spoke grid $K_{1,m} \square K_{1,n}$: row relays r_i and column relays c_j report to the central controller $a_{0,0}$, while intersection sensors are represented by the vertices $a_{i,j}$.
- Only a selected set W of sensors is upgraded to **landmarks** such as stronger radios, wired back-haul nodes, or reliable beacons; the remaining sensors stay low-power.
- Each sensor is identified by its hop-count vector to the landmarks:

$$r(a_{i,j} \mid W) = \big(d(a_{i,j}, w_1), \ldots, d(a_{i,j}, w_{|W|})\big).$$

- If W is a metric basis, then these vectors are all distinct. Hence $\dim(K_{1,m} \square K_{1,n})$ is exactly the minimum number of upgraded nodes needed for unambiguous localization.

Why consider $F_m \square K_n$?

From relay grids to clustered networks

- The product $K_{1,m} \square K_{1,n}$ models star-based relay layouts, where communication is organized through central relays.
- In many wireless sensor networks and cluster-based communication systems, a local cluster is richer than a pure star: besides a common hub, short local links provide redundancy inside the cluster.
- The friendship graph F_m captures this local motif: it consists of m triangles sharing a common hub. Thus it keeps the hub-and-spoke structure, but adds local triangular connectivity.

Why consider $F_m \square K_n$?

Cluster-backbone interpretation

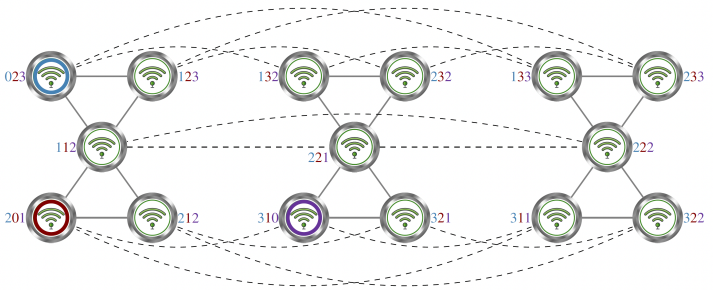
- A vertex (u, j) of $F_m \square K_n$ represents the local position $u \in V(F_m)$ in the j-th cluster or layer.
- Each layer is a copy of F_m, while the factor K_n represents a dense inter-cluster backbone. In particular,

$$d_{F_m \square K_n}\big((u, j), (v, k)\big) = d_{F_m}(u, v) + \mathbf{1}_{\{j \neq k\}}.$$

- Therefore, determining $\dim(F_m \square K_n)$ gives the optimal landmark budget for localization, monitoring, and fault detection in this clustered architecture.

$\dim(F_m \Box K_n)$

Motivated by **wireless sensor networks** and **cluster-based communication systems**, we study the metric dimension of the Cartesian product $F_m \Box K_n$.



Exact Results for $\dim(F_m \square K_n)$

Assume throughout that $m, n \geq 2$.

Theorem (Davoodi and Jannesari, 2026)

If $n - 1 \geq 2m$, *then:*

$$\dim(F_m \square K_n) = n - 1$$

Theorem (Davoodi and Jannesari, 2026)

If $\frac{m}{2} < n - 1 < 2m$,

$$\dim(F_m \square K_n) = \begin{cases} 5, & \textit{if } (m,n) = (3,4), \\ \left\lceil \dfrac{2(m+n-1)}{3} \right\rceil, & \textit{otherwise.} \end{cases}$$

Theorem (Davoodi and Jannesari, 2026)

If $\frac{m}{2} \geq n - 1$, *then:*

$$\dim(F_m \square K_n) = m$$

Intermediate regime $\frac{m}{2} < n - 1 < 2m$: why it is difficult

In the two extremal regimes, either the number of columns or the number of triangle pairs dominates the construction. In the intermediate regime, neither parameter alone controls the metric dimension; row and column constraints interact simultaneously.

Structural reduction

Using a sequence of exchange arguments, one may choose a metric basis B with the following normalized form:

$$B \cap \{c_1, \ldots, c_n\} = \emptyset, \qquad B \cap C_n = \emptyset, \qquad B \cap R_i' = \emptyset \quad \text{for all } i.$$

Moreover, after deleting the empty column C_n, the basis must satisfy

$$B \cap R_i \neq \emptyset, \qquad B \cap C_j \neq \emptyset, \qquad |B \cap (R_i \cup C_j)| \geq 2$$

for every $1 \leq i \leq m$ and $1 \leq j \leq \ell := n - 1$.

Reduction to a matrix optimization problem

Let $\ell := n - 1$. The normalized basis can be encoded by a binary matrix

$$X = (x_{i,j}) \in \{0,1\}^{m \times \ell}, \qquad x_{i,j} = 1 \Longleftrightarrow a_{i,j} \in B.$$

Thus $|B|$ is the number of 1-entries of X.

The resulting ILP

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} \sum_{j=1}^{\ell} x_{i,j} \\ \text{subject to} \quad & \sum_{j=1}^{\ell} x_{i,j} \geq 1, \qquad 1 \leq i \leq m, \\ & \sum_{i=1}^{m} x_{i,j} \geq 1, \qquad 1 \leq j \leq \ell, \\ & \sum_{q=1}^{\ell} x_{i,q} + \sum_{p=1}^{m} x_{p,j} - x_{i,j} \geq 2, \qquad 1 \leq i \leq m,\ 1 \leq j \leq \ell. \end{aligned}$$

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Theorem(Matrix lower bound) [Davoodi and Jannesari, 2026]

Every feasible matrix has at least

$$\left\lceil \frac{2(m+\ell)}{3} \right\rceil = \left\lceil \frac{2(m+n-1)}{3} \right\rceil$$

one-entries. Hence every metric basis has at least this many vertices.

Set-system form and extremal graph theory

Theorem (Set-system bound; Davoodi and Jannesari, 2026)

Let $m, \ell \in \mathbb{N}$, and let $\mathcal{A} = \{A_1, \ldots, A_m\}$, and $\mathcal{B} = \{B_1, \ldots, B_\ell\}$ be two partitions of a finite set U. If for every $1 \le i \le m$, $1 \le j \le \ell$, $|A_i \cup B_j| \ge 2$, then

$$|U| \ge \left\lceil \frac{2(m+\ell)}{3} \right\rceil .$$

Bipartite graph interpretation

The proof views U as the edge set of a bipartite graph $[A, B]$ each element $u \in U$ gives one edge between the two parts containing it. Thus $|U| = |E|$. The union condition excludes a component consisting of a single edge, and a component-wise counting argument gives $|E| \ge 2(m + \ell)/3$.

Intermediate regime: what about the upper bound?

Why it works

The chosen set W is arranged so that

- every row and every column is represented, and
- if a column contains only one landmark, then that landmark lies in a row containing two landmarks.

By some developed lemmas, these properties are sufficient to imply that W resolves $F_m \Box K_n$.

Intermediate regime: what about the upper bound?

	vertical part			staircase part			
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
R_1	●	○	○	○	○	○	○
R_2	●	○	○	○	○	○	○
R_3	○	●	○	○	○	○	○
R_4	○	●	○	○	○	○	○
R_5	○	○	●	○	○	○	○
R_6	○	○	●	○	○	○	○
R_7	○	○	○	●	●	○	○
R_8	○	○	○	○	○	●	●

Hence, for the intermediate regime, $\dim(F_m \square K_n) = \left\lceil \frac{2(m+n-1)}{3} \right\rceil$.

Thank you for your attention.