Total colouring of cubic Halin graphs

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Basic notions

Graph $G = (V, E)$ - simple, undirected

An **edge colouring** of a graph G is a map $c' : E(G) \rightarrow M$. If $c'(uv) \neq c'(vw)$ for all $u, v, w \in V(G)$ such that $uv, vw \in E(G)$, we say that the edge-colouring is **proper**. The smallest $k = |M|$ for which G has a proper edge colouring is said to be the **chromatic index** $\chi'(G)$ of G.

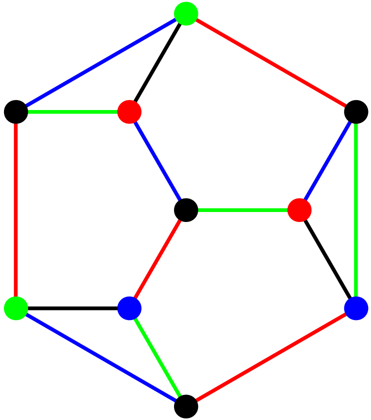
An **vertex colouring** of a graph G is a map $c : V(G) \rightarrow M$. If $c(u) \neq c(v)$ for all $uv \in E(G)$, we say that the vertex colouring is **proper**. The smallest $k = |M|$ for which G has a proper edge-colouring is said to be the **chromatic number** $\chi(G)$ of G.

Total colouring

Let $c' : E(G) \to M$ be a proper edge colouring and let $c : V(G) \to M$ be a proper vertex colouring such that $c(v) \in S^C(v)$ for each vertex $v \in V(G)$. Then $c'' : V(G) \cup E(G) \to M$ is a **total colouring** of G if

$$c''(x) = \begin{cases} c(x), \text{ if } x \in V(G), \\ c'(x), \text{ if } x \in E(G). \end{cases}$$

The smallest $k = |M|$ for which G has a total colouring is called the total chromatic index $\chi''(G)$.



Known results

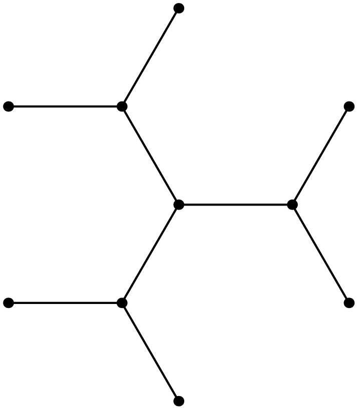
Theorem (Feng, Lin)

Let G be a subcubic graph G. Then

$$\chi''(G) \leq 5.$$

Halin graphs

A graph H is **Halin** if it can be obtained from a plane embedding of a tree T with at least three leaves by adding a cycle passing through all the leaves with no crossing edges. Every Halin graph is connected and planar.



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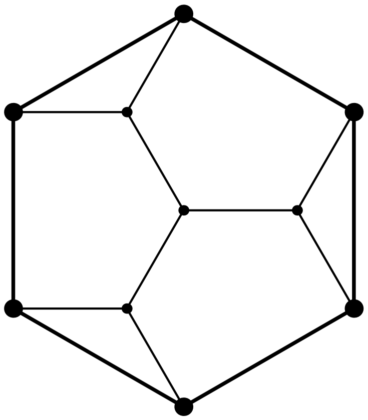

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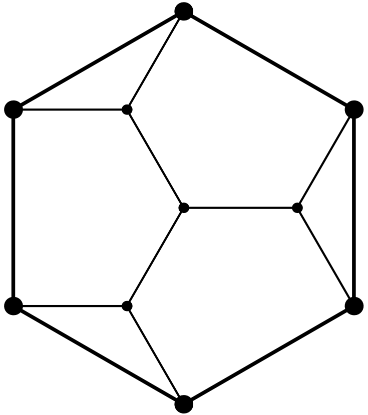
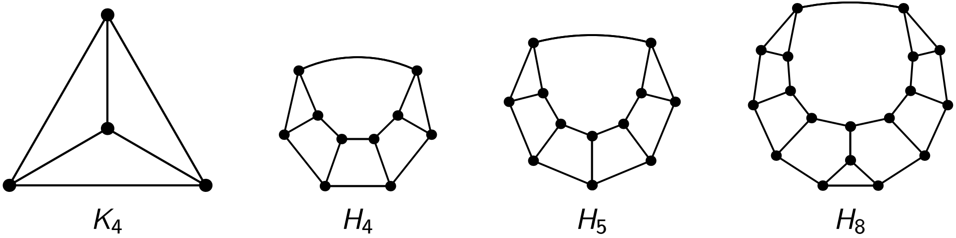
A vertex corresponding to a leaf in T is called **peripheral** or **ring** vertex in H. A vertex which is not peripheral is **spanning**.

An edge connecting two peripheral vertices in H is called **peripheral**. An edge which is not peripheral is called **spanning**.



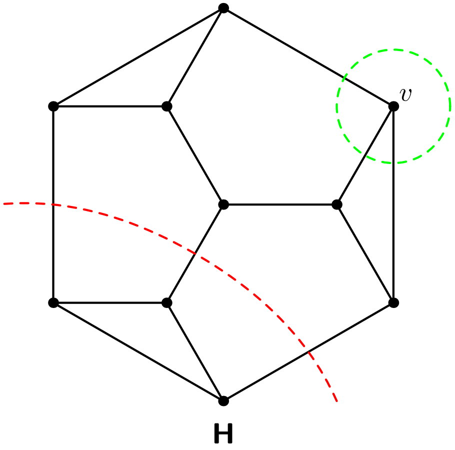
Cubic Halin graphs with $\chi'' > 4$



Question

What is the number of cubic Halin graphs with $\chi'' = 5$?

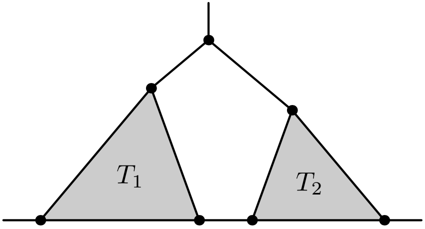
Let X, Y be a partition of the vertex set of a cubic Halin graph H such that $E(X, Y)$ contains precisely 1 spanning and 2 peripheral edges. Then X (or Y) is a **Halin tripole**.
If $Y = \{v\}$ contains exactly one peripheral vertex of H, then the tripole X can be denoted as T_v^H.
The **rank** $r(T)$ of a tripole T is the number of spanning vertices in T.



Halin tripole decomposition

Observation

A cubic Halin tripole T of rank $n+1$ can be decomposed into tripoles T_1 and T_2 such that $r(T_1) + r(T_2) = n$.

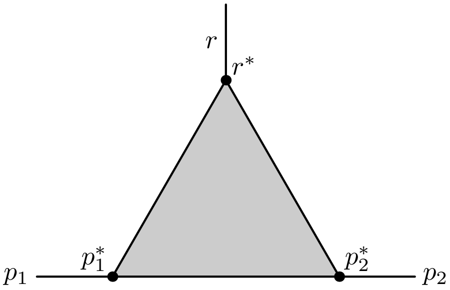


Palettes

Let T be a cubic Halin tripole and let $\{r, p_1, p_2\}$ be its semi-edges and c'' be a total 4-colouring of T. Then the sextuple

$$(c''(r), c''(r^*), c''(p_1), c''(p_1^*), c''(p_2), c''(p_2^*),)$$

is said to be an **extendable colouring** of T.



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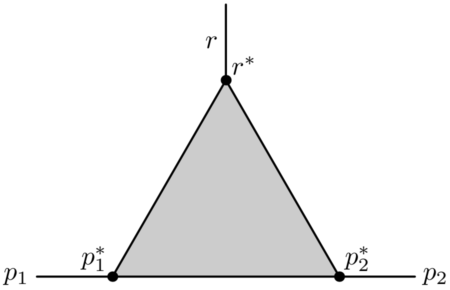
Let $P \subseteq \{0, 1, 2, 3\}^6$ be a set of sextuples. Then P is said to be a **palette**. If P is a complete set of extendable colourings of a tripole T, then P is its palette, denoted as $\mathcal{P}(T)$.



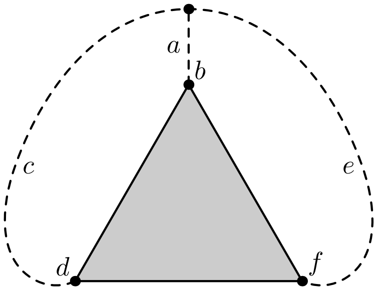
Completable palette

We say that a sextuple (a, b, c, d, e, f) is **completable** if

(i) $|\{a, c, e\}| = 3$, and

(ii) $|\{a, b, c, d, e, f\}| = 3$.

A palette is said to be **completable** if it contains a completable sextuple.



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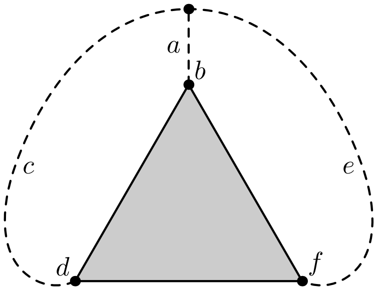
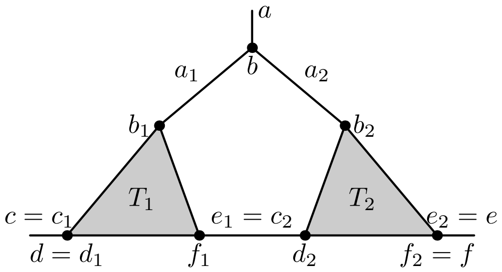


Lemma

Let H be a cubic Halin graph and v its peripheral vertex. Then H is total 4-colourable if and only if $\mathcal{P}(T_v^H)$ is completable.

Induced expandable colourings



We say that sextuples $(a_1, b_1, c_1, d_1, e_1, f_1)$ and $(a_2, b_2, c_2, d_2, e_2, f_2)$ are **composable** if

(i) $e_1 = c_2$,

(ii) $|\{e_1, f_1, c_2, d_2\}| = 3$,

(iii) $a_1 \neq a_2$ and

(iv) $|\{a_1, b_1, a_2, b_2\}| \leq 3$.

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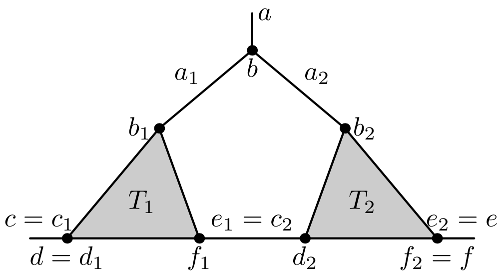

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(iv) $|\{a_1, b_1, a_2, b_2\}| \leq 3$.

Lemma

Let T be a Halin tripole decomposable into T_1 and T_2. Then $(a, b, c, d, e, f) \in \mathcal{P}(T)$ if and only if there exist $(a_1, b_1, c, d, e_1, f_1) \in \mathcal{P}(T_1)$ and $(a_2, b_2, c_2, d_2, e, f) \in \mathcal{P}(T_2)$ which are composable.

Linear-time decision algorithm

Theorem (Kardoš, M, 2026+)

The decision problem of total 4-colouring of cubic Halin graphs has an algorithm with linear-time complexity.

Strategy

Let P_0 be the palette of the Halin tripole of rank 0. A set of palettes $\mathcal{P}$ is **closed under composition** if for any two palettes $P_x, P_y \in \mathcal{P}$ we have $P_x \oplus P_y \in \mathcal{P}$. Let $\mathcal{P}^+$ be the inclusion-wise minimal set of palettes containing P_0, closed under composition. Clearly, P^+ is the set of all realisable palettes.

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Task

Enumerate $\mathcal{P}^+$.

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Analyse the $\mathcal{P}^+$

Total colouring of cubic Halin graphs

Theorem (Kardoš, M, 2026+)

Let H be a cubic Halin graph other than K_4, H_4, H_5 and H_8. Then

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$\implies$ *Proof of the Theorem.*

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https://arxiv.org/abs/2603.23189

On efficient composition

Observation

Let P be a palette and let $\sigma \in S_4$ be a permutation. If $(a, b, c, d, e, f) \in P$ then $(\sigma(a), \sigma(b), \sigma(c), \sigma(d), \sigma(e), \sigma(f))$.

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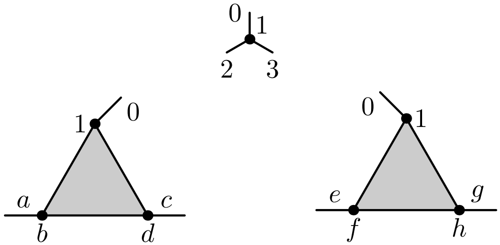
Definition

Let P be a palette. We say that $\Phi(P)$ is the *fixed palette* of P iff

$$\Phi(P) = \{(a, b, c, d) : (0, 1, a, b, c, d) \in P\}.$$

On efficient (?) composition



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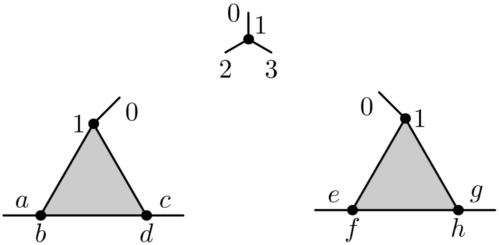
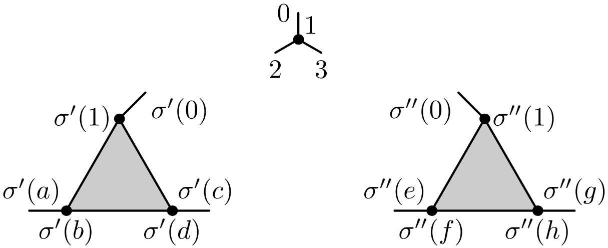


Definition

Let $\Phi(P)$ be the fixed palette of a palette P. Let $S' \subseteq S_4$ be an arbitrary set of permutations. We say that $\Phi_{S'}(P)$ is the *recolouration* of Φ iff

$$\Phi_{S'}(P) = \{(\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \mid \sigma \in S', (a, b, c, d) \in \Phi(P)\}.$$

On efficient (??) composition



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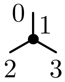

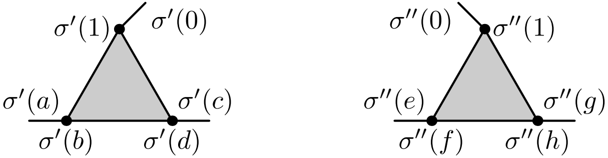

Lemma

Let T be a cubic Halin tripole decomposable into tripoles T_1 and T_2. Let $P = \mathcal{P}(T)$, $P_1 = \mathcal{P}(T_1)$ and $P_2 = \mathcal{P}(T_2)$ be palettes of tripoles T, T_1 and T_2. Then

$$\Phi(P) = (\Phi_{S'}(P_1) \oplus_f \Phi_{S''}(P_2)) \cup (\Phi_{S''}(P_1) \oplus_f \Phi_{S'}(P_2)),$$

where $S' = \{\sigma' \mid \sigma'(0) = 2, \sigma'(1) \notin \{1, 2\}\}$ and $S'' = \{\sigma'' \mid \sigma''(0) = 3, \sigma''(1) \notin \{1, 3\}\}$.

Characteristic matrix

Definition

Let $M_{16\times 16}$ be a matrix over the field $(\{0,1\},\vee,\wedge)$. We say that $M=\mathcal{M}(\Phi)$ is the **characteristic matrix** of a fixed palette Φ if for each quadruple (a,b,c,d):

$$(a,b,c,d)\in\Phi \Leftrightarrow M_{i,j}=1,$$

where $i=4a+b$ and $j=4c+d$.

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$$(a,b,c,d)\in\Phi \quad \Longrightarrow \quad \begin{array}{cc} \begin{bmatrix} M_{0,0} & \cdots & M_{15,0} \\ \vdots & 1 & \vdots \\ M_{0,15} & \cdots & M_{15,15} \end{bmatrix} & 4c+d \\ 4a+b & \end{array}$$

Palette union

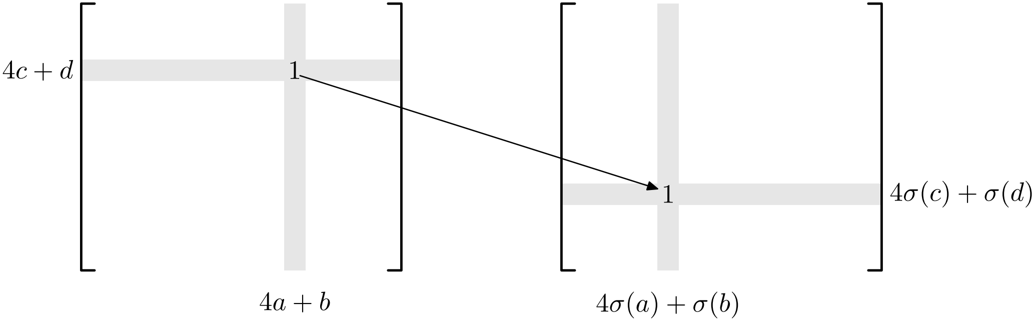
Lemma

Let P_1 and P_2 be fixed palettes. Then

$$(a, b, c, d) \in \Phi_1 \cup \Phi_2 \Leftrightarrow (\mathcal{M}(\Phi_1) \vee \mathcal{M}(\Phi_2))_{i,j} = 1,$$

for $i = 4a + b$ and $j = 4c + d$.

Palette recolouration



$$(a, b, c, d) \in \Phi \implies (\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \in \Phi$$

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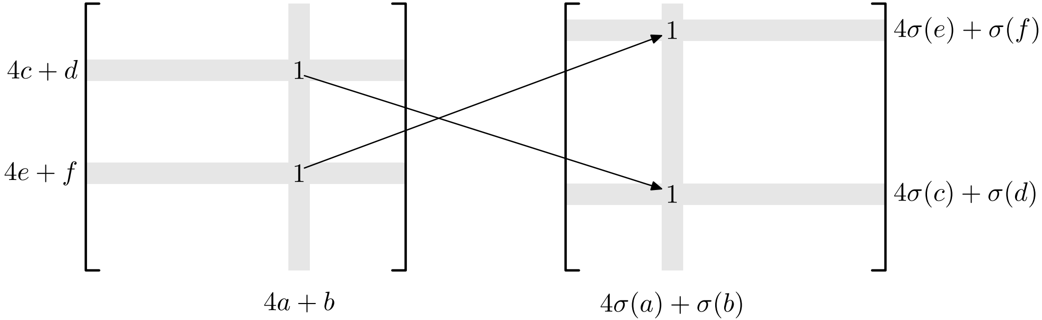

$$(a, b, c, d) \in \Phi \implies (\sigma(a), \sigma(b), \sigma(c), \sigma(d)) \in \Phi$$

$$(a, b, e, f) \in \Phi \implies (\sigma(a), \sigma(b), \sigma(e), \sigma(f)) \in \Phi$$

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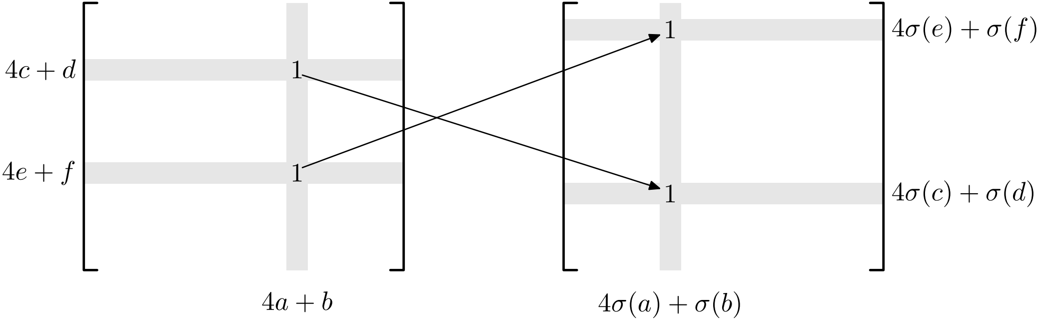
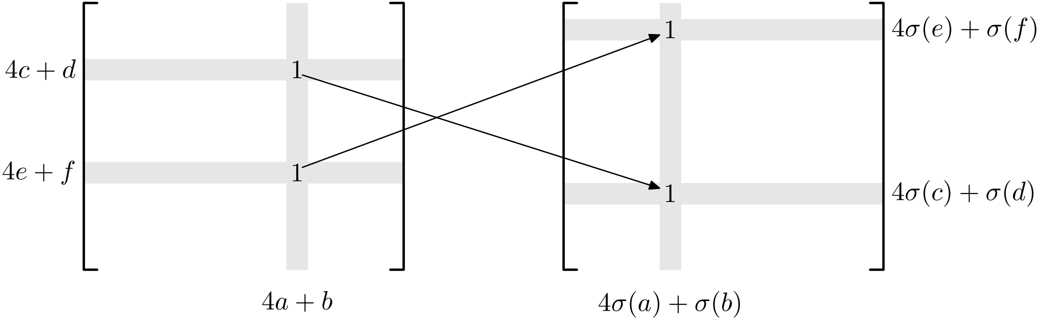


$\varphi_\sigma : \{0,1\}^{16} \to \{0,1\}^{16}, |\varphi_\sigma| = 2^{16}$

Palette recolouration



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Memory needed (per permutation): **2B** $*2^{16}$ = **2B**$*$65336 $\approx$ 130**KB**

Palette recolouration

Let $\sigma \in S_4$ be a permutation. Let $\varphi_\sigma : \{0,1\}^{16} \to \{0,1\}^{16}$ be a be function on the set of binary vectors of length 16. Let $v \in \mathbb{Z}_2^{16}$ be a vector. We say that φ_σ is the **characteristic vector permutation** if for each $4a + b = i \in \{0, \ldots, 15\}$:

$$v_i = 1 \Leftrightarrow \varphi(v)_{\sigma(i)} = 1.$$

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Lemma

Let P be a palette. Let Φ be the fixed and let $\Phi_{S'}$ be its recolouration $S' \subseteq S_4$. Let φ be the characteristic vector permutation and let $M = \mathcal{M}(\Phi)$. Then

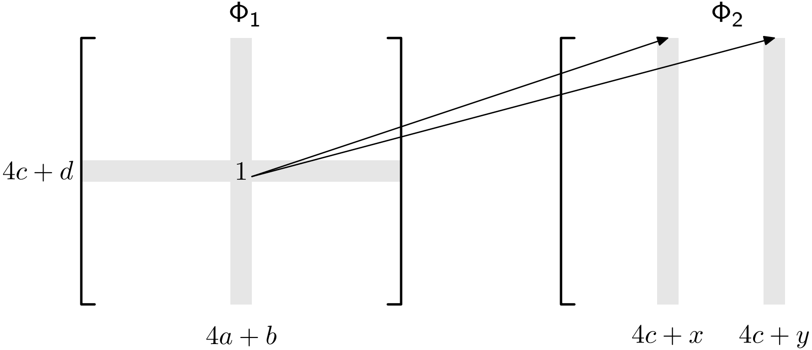
$$\rho_{S'}(\mathcal{M}(\Phi)) = \mathcal{M}(\Phi_{S'}) = \bigvee_{\sigma \in S'} (\varphi_\sigma(M_{\sigma^{-1}(0)}), \dots, \varphi_\sigma(M_{\sigma^{-1}(15)})).$$

Palette composition

$$\Phi_1 \qquad\qquad \Phi_2$$

$$\begin{array}{cc} 4c+d & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \\ & \quad 4a+b \end{array} \qquad\qquad \begin{array}{c} \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \\ 4c+x \quad 4c+y \end{array}$$

Palette composition



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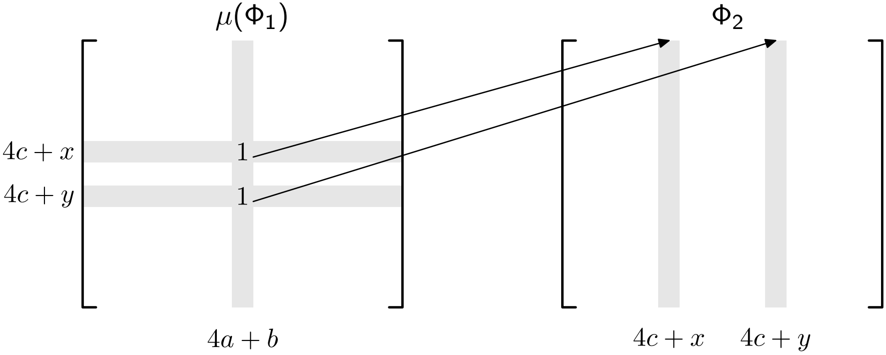

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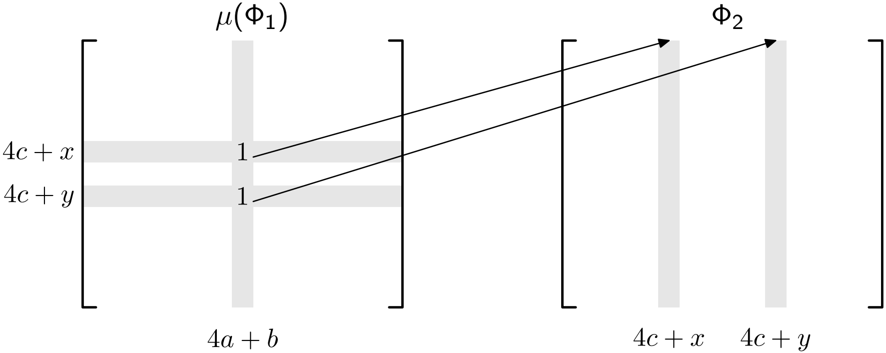

$\mu : \{0,1\}^{16} \to \{0,1\}^{16}$, $|\mu| = 2^{16}$

Again: $\approx$ 130 **KB** to store it precomputed

Palette composition

$$\Phi \qquad\qquad\qquad \Phi_2$$

$$\underset{4a+b}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}} \qquad \begin{array}{c c} \begin{array}{c} 4e+f \\ 4g+h \\ 4i+j \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ & \begin{array}{cc} 4c+x & 4c+y \end{array} \end{array}$$

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$(a, b, c, d) \in \Phi_1, (c, x, e, f), (c, y, g, h) \in \Phi_2 \implies$

Palette composition

$$\begin{array}{ccc} \Phi & & \Phi_2 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{matrix} 4e+f \\ 4g+h \\ 4i+j \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ 4a+b & & 4c+x \quad 4c+y \end{array}$$

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$(a,b,c,d) \in \Phi_1, (c,x,e,f), (c,y,g,h) \in \Phi_2 \implies$

$\implies (a,b,e,f), (a,b,g,h) \in \Phi$

Composition: essentially multiplication of a binary matrix

Palette composition

Definition

Let $v \in \{0,1\}^{16}$ be a binary vector, let $4a + b$ be a number, and let $x, y \in \{0,1,2,3\} \setminus \{a,b\}$ be two distinct numbers. Let $\mu : \{0,1\}^{16} \to \{0,1\}^{16}$ be a **pairing** function defined as

$$v_{4a+b} = 1 \Leftrightarrow [\mu(v)_{4a+x} = 1 \land \mu(v)_{4a+y} = 1].$$

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$$v_{4a+b} = 1 \Leftrightarrow [\mu(v)_{4a+x} = 1 \wedge \mu(v)_{4a+y} = 1].$$

Lemma

Let Φ, Φ_1 and Φ_2 be fixed palettes such that $\Phi = \Phi_1 \oplus_f \Phi_2$. Then

$$\mathcal{M}(\Phi) = \mu(\rho_{S_1'}(\mathcal{M}(\Phi_1)))\ \rho_{S_2'}(\mathcal{M}(\Phi_2)).$$

The End

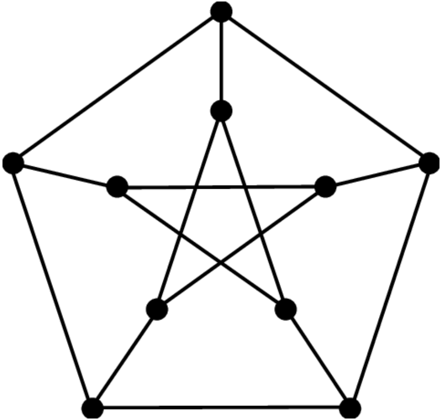

Figure: Picture of the Petersen graph serving the purpose of checking all the boxes